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**A Monte Carlo study of new time series statistical tests and their application to the modeling of price dynamics in futures markets**

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A MONTE CARLO STUDY OF NEW TIME-SERIES STATISTICAL TESTS
AND THEIR APPLICATION TO THE MODELING OF
PRICE DYNAMICS IN FUTURES MARKETS

by

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# DEDICATION

**To my teachers, classmates, and family**

# A MONTE CARLO STUDY OF NEW TIME-SERIES STATISTICAL TESTS AND THEIR APPLICATION TO THE MODELING OF PRICE DYNAMICS IN FUTURES MARKETS

BY

HONG GAO

## ABSTRACT

Modeling price dynamics in financial markets has become an important research area in financial economics. In the past empirical studies of financial price movements were based on methods that were incapable of detecting or modeling nonlinear serial dependence that characterizes financial market data. Recently, advances in the study of nonlinear dynamics in the physical sciences have motivated researchers to apply nonlinear time-series models to the study of financial and economic data. This dissertation investigates three statistical tests which can detect nonlinear serial dependence, and applies these tests and two nonlinear time-series models to futures markets.

In this dissertation, the finite sample properties of the BDS, TAR-F, and $Q^2$ test are evaluated using Monte Carlo experiments. Monte Carlo findings show that the finite sample distribution of the tests under the data generating processes (DGPs) of the null hypothesis approximates their asymptotic counterpart quite closely. The power of the tests on DGPs of alternative hypotheses reaches unity at sample size of 1000 when the DGPs are not too close to the DGP of the null hypothesis.

Base on findings of Monte Carlo investigation, the three tests and two nonlinear time-series models are applied to the study of price dynamics in futures markets. The futures studied are the S&P 500, Crude Oil, Japanese Yen, Deutsche Mark, and Eurodollar futures. The results show that the price changes of all five futures have nonlinear serial dependence, and that they can be modeled by nonlinear time-series models, either GARCH, or TAR, or combined TAR-GARCH model.

The main conclusions to emerge from the findings of this dissertation are as follows. The three tests are reliable for detecting serial dependence, including nonlinear serial dependence. The tests work well when sample size equals 1000 or larger and the sample's departure from the null hypothesis is not too small. When analyzing futures prices, we have to account for nonlinear serial dependence, use nonlinear models with conditional heteroskedasticity and conditional mean change.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Background

Financial economics has become a major field of economic analysis and much recent empirical work has focused on econometric modeling of price dynamics in financial markets.[1] The study of price dynamics in financial markets may seem little more than of academic interest, but price changes in financial markets directly affect economic behavior. Price changes in financial markets, for example, affect consumers' consumption and savings behavior. As another example, producers' production planning and investment decisions regarding equipment, structures, and growth, and financial traders' strategies all depend fundamentally on price dynamics. Finally, price changes in financial markets also affect government economic policy. Clearly, this dissertation's study of price dynamics in financial markets is central to understanding economic behavior in financial economics.

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[1] See Eatwell, Milgate, and Newman (1989), Ross (1989). While some such as Taylor (1986) use the term "financial market" for the market of stock, commodities, and currencies, others may include the bond and T-note. In this dissertation I use the financial market to refer to markets of stock, bond, T-note, futures (commodities, stock index, currencies, T-bond and T-note futures), options, and currencies.

Futures markets are good examples of financial markets and they have drawn much interest by researchers. Futures market participants are economic agents who seek information on future commodity prices, economic agents who want to avoid unwanted risk, and economic agents who want to make speculative profits. By providing people with information on future commodity prices, futures markets assist economic agents in making optimal consumption, investment and production decisions. By bridging unwanted risk from producers (hedgers) to speculators, futures markets also help improve efficiency in conducting economic activities. Futures markets also provide liquidity to the hedgers by the speculators.

Price dynamics in financial markets are believed to be governed by the efficient market hypothesis (EMH). It states that market price captures all available information. One version of the EMH is the weak-form efficient market hypothesis. If the market is weak-form efficient, then current asset prices reflect all information contained in historical prices. In this case a trader could not, on average, make economic profit in the market based on price information. The general testing method of the efficient market hypothesis is based on price forecast errors. Unfortunately, this is a joint test of the underlying forecast model and of the market efficiency hypothesis[2]. The standard testing techniques for market efficiency involve autocorrelations and autocovaraince[3]. These techniques have been shown to be incapable of detecting some nonlinear processes

[2] See Fama (1976), Copeland and Weston (1988), and Heimstra (1990).

[3] See Fama (1975, 1976), Miskin (1978), Hoffman et. al. (1984), and Summers (1986).

(Barnett and Chen, 1989). Thus, if price dynamics in financial markets are inherently nonlinear, then standard tests may yield misleading and perhaps incorrect results regarding the efficient market hypothesis.

Perhaps the earliest model of price dynamics in financial markets was the random walk model proposed by Bachelier (1900). Extension of the random model lead to development of the martingale model and the mean-reverting model. All these models are consistent with the weak-form EMH. The random walk model assumes that the time series of asset price changes are independently and identically distributed (IID) Gaussian. Other types of random walk models, in the form of log-price changes and non-Gaussian distributions, have also been developed by other researchers[4]. Some recent empirical works have questioned the validity of the random walk model. Lo and Mackinlay (1988), for instance, applied a variance-ratio test to stock returns and rejected the random walk hypothesis.

The martingale model was formulated by Samuelson (1965) for study of price dynamics in financial markets. It requires the expected value of price changes to be zero. But unlike the random walk model, Samuelson's martingale model does not require the time series of price changes to be independent nor does it require that price changes to be identically distributed. In general, the test of the martingale model, like the test of efficient market hypothesis, also depends on econometric model used to predict prices. Obviously, before we can formulate any predicative model and test the validity of the

[4] Samuelson (1955), Osborne(1959), and Alexander(1961) suggested log price changes; Mandelbrot (1963), Fama (1969), and So (1987) suggested Stable Paretian distribution; Clark (1973) suggested sub Gaussian distribution.

martingale model, we have to identify in a statistical sense whether prices follow a linear or nonlinear process. This presents the financial market analysis with an econometric model specification problem.

The model of mean-reverting processes suggests that price dynamics in financial markets are the sum of random walks and a stationary mean-reverting process.[5] According to this model, if market prices deviate from their mean (perhaps the fundamental value) beyond some range, prices revert to the mean, and prices will be negatively correlated with past prices. Fama and French (1988) found the negative autocorrelation in stock prices over long horizons which supports the mean-reverting process. But some recent studies reexamined the mean-reverting process and pointed out, however, that when stock returns from certain periods are excluded, the evidence of mean-reverting disappeared. Lo and Mackinlay's (1988) study, based on variance-ratio statistic, rejects the mean-reverting process model as well as the random walk model.

Our review of the literature has shown that some empirical studies have rejected financial price models, the use of the random walk, the martingale, and the mean-reverting as models of financial price movements. Since there appears to be a model specification problem with the popular models, the test statistics associated with these models, and the testing method for efficient market hypothesis in financial markets (such as autocorrelation function and variance ratio test), may fail to detect serial dependence in financial prices. More fundamentally, perhaps the popular models are markedly incapable of modeling price dynamics in financial markets. Naturally, in this dissertation

[5] See Shiller and Perron (1985), Poterba and Summers (1988).

we examine other models and methods, such as nonlinear models and nonlinear techniques, and assess whether they provide a more reliable way to model and test price dynamics in financial markets.

## 1.2 Motivation

Inspired by the success of nonlinear dynamic analysis in the physical sciences, researchers in economics and finance started to apply nonlinear models in their research. The overlapping generation model of utility maximization by Benhabib, Day, and Grandmont (Grandmont, 1985), for example, is a nonlinear dynamic model. Savit (1989) also used a nonlinear deterministic chaotic model in an option-price model. Aiyagari, Eckstein, and Eichenbaum (1985) derived a switching stochastic model for prices of storable goods. Hsieh (1988) developed a nonlinear stochastic rational expectation model of exchange rates under central bank interventions. Finally, Lai and Pauly (1988) formulated a model for foreign exchange rates with a varying conditional variance. These theoretical studies of nonlinear dynamic models in economics and finance are the motivations for my study and application of nonlinear statistical methods to futures prices. To see why this is so consider the following.

Many findings based on the study of nonlinear dynamics in the physical sciences are very relevant to financial economics. For example, a nonlinear system can generate a deterministic chaotic time sequence that appears to be random.[6] And it is well known that price movements in financial markets appear random. Physical sciences' findings also revealed how a nonlinear system can evolve from the motion of regularity to the motion of chaos (irregular or unpredictable), and found evidence characteristic of a chaotic time sequence. But most importantly, the physical sciences developed methods that can be applied to analyze any nonlinear system. More specifically, physicists showed how to study the properties of a complex multi-dimensional dynamic system based on a single observed variable.[7] They tell us that, for instance, if a dynamic system is governed by a two-dimensional equation with two variables, we can derive the properties of the system from one observed variable. This result is fundamentally important in our study, because if other endogenous variables also enter the system in determining financial prices, then without knowing these "other" variables, we can still investigate the properties of the system just by using the price series alone. The rich features of nonlinear model and the foundations for analyzing nonlinear models found in physical sciences further motivate us to use nonlinear models in economics and finance which is the theme of this dissertation.

Accompanying the studies in nonlinear dynamic system and deterministic chaos has been the development of methods used to detect a chaotic time sequence (which looks random but is not random). These methods, however, do not give us test statistics.

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[6] See May (1975), Ott (1981), Barnett and Chen (1989), Baumol and Benhabib (1989).

[7] This is due to a theorem by Takens (1983).

Furthermore, they are not intended to deal with stochastic time-series processes. And some empirical studies in finance are leaning towards nonlinear stochastic model rather than nonlinear deterministic chaotic model. So recently some statistical tests of nonlinearity, which had not been used much in the past, have started to attract new interest in econometrics and financial markets. Moreover, a new statistical test has been developed with techniques from the physical sciences. The major part of this dissertation is exactly focused on the statistical tests that can detect nonlinearity.

Three statistical tests investigated in this dissertation are the BDS, TAR-F, and $Q^2$ test. The BDS statistic is based on correlation integral, and is designed to test whether a time series is independently and identically distributed (IID). One major advantage of the BDS statistic is that it can detect nonlinear serial dependence in time series. This is a situation where the traditional technique of autocorrelation and autocovariance fails because they can not detect nonlinear serial dependence. When the BDS is applied to the linearly fitted data, it can be used to detect nonlinearity. The BDS statistic can be also applied to the residuals of the forecasting model for testing the adequacy of the model. The TAR-F test can also detect nonlinear time series process. And the $Q^2$ test can detect autocorrelation in the squared values of time series. The $Q^2$ test is especially useful for detecting autoregressive conditional variance in time series where the linear technique of autoregressive test fails. These interesting properties of the three tests are primarily the reasons for these tests being selected to be studied in this dissertation.

The next step in study of financial prices after the detecting of nonlinear dependence is the modeling of nonlinear dependence in financial prices. Two nonlinear econometric models are relevant to the price changes in financial markets: the threshold autoregressive (TAR) model and the generalized autoregressive conditional heteroskedasticity (GARCH) model. The two models are employed in this dissertation for modeling futures prices. The TAR model of Tong (1978) represents a model which has nonlinearity in conditional mean. The applications of the TAR model has shown that it can produce features such as limit-cycle, amplitude dependent frequencies, and jump phenomena. All these features have been observed in economics and finance. The TAR process can model the asymmetric and periodic behavior seen, for example, in sunspot data and Canadian LYNX data.[8] The GARCH model of Bellerslev (1986) assumes that the current value of the time series conditioned on past information has a normal distribution, where the variance of the distribution is an autoregressive process on both of the past values and of past variances. This feature is especially useful for analyzing price volatility in financial markets.

## 1.3 Objectives

The discussion in last section shows the motivations for study nonlinear statistical tests, and the motivations for applying these tests and the two nonlinear econometric models to futures prices. This section sets the objectives of the dissertation. They are: a) to investigate the finite sample properties of three test statistics (the BDS statistic, the TAR-statistic, and the $Q^2$ statistic) using the Monte Carlo simulation and Hendry's (1984)

[8] See Tong and Lim (1980), and Tsay (1989).

response surface methodology; b) to apply the three tests in analyzing of price changes in futures markets; c) to use two nonlinear models (TAR and GARCH model) for modeling price changes in futures markets.

The three statistical tests discussed in last section have the ability to detect nonlinear serial dependence in time series data. But their exact finite sample distribution can not be derived. Thus the asymptotic distribution of the tests is used in the empirical analysis of economic and financial time series data. In this case it is important for researchers to know how does the finite sample distribution compare to the asymptotic distribution. Therefore the first objective of this dissertation is to learn the finite sample properties of the three statistical tests using Monte Carlo experiments. Specifically this dissertation will answer how well do the finite sample results of the tests approximate the asymptotical results under the null hypotheses of the tests. This dissertation will also investigate how rapidly does the power (i.e., the rejection frequency of the null hypothesis) of the tests approach unity under different circumstances when the alternative hypotheses of the tests are true.

The other objective of this dissertation is to apply the three tests to time series of futures prices. In the process, I will show what will be lost by using old testing method of autocorrelation and what will be gained by using the new tests. Then I will test whether futures prices have serial dependence, whether the serial dependence is linear or nonlinear. I will also demonstrate how these new statistical tests can be used for detecting, identifying and modeling of nonlinear serial dependence in time series data.

The final objective of this dissertation is to apply the TAR model and the GARCH model to futures prices if the tests indicate such nonlinearity exists in futures prices. The models will be examined to see whether they can describe price movements in futures markets. The relation of the estimated nonlinear models with the financial theories and models will also be discussed.

## 1.4 Methodology

To accomplish the objectives of this dissertation, we employ the following methods: a) Monte Carlo experiment for deriving the finite sample properties of the three tests, i.e., the BDS test, the TAR-F test, and the $Q^2$ test; b) calculation of the three test statistics which are needed for Monte Carlo investigation of the finite sample properties of the tests and for application of the three tests to futures prices; c) estimation of the two econometric nonlinear time series models, the TAR model and the GARCH model, for modeling of futures prices.

In order to study the finite sample properties of the three test statistics, Monte Carlo method is used. The Monte Carlo method is a well known simulation procedure. It is used primarily for solving a variety of problems which are too difficult to solve analytically.[9] As the example, because the exact finite sample distribution of the three test statistics can not be derive analytically, we employ Monte Carlo methods to investigate the finite-sample performance of the tests based on an empirical finite-sample

[9] The are two major applications of Monte Carlo method. One is for calculating results of complicated integration. The other is for investigating the statistical properties of estimate or test statistic. The later is used in this dissertation.

density. The empirical finite-sample density is based on simulated time-series. These simulated time-series can be independently and identically distributed (IID), or have linear or nonlinear serial dependence. The means, the standard deviations, and the rejection frequencies of the test statistics can be calculated from the replicated statistics on the simulated samples and compared with asymptotic results. In this manner we can assess how well the finite sample distribution approximates the asymptotic distribution. The means, the standard deviations, and the rejection frequencies of the test statistics are also regressed on the sample size and the parameters of the DGP to obtain the response surfaces of the test statistics. The response surface provides a statistical model that summarizes the properties of a tests under different values of the DGP parameters and under different sample sizes.

The BDS statistic is calculated from the correlation integral of the time series. Under the independently and identically distributed (IID) null hypothesis, the BDS statistic is asymptotically standard normal distributed. The TAR-F statistic is calculated from the recursive estimation of the arranged least square regression of the time series. Under the null hypothesis that the time series is linear, the TAR-F statistic has an F-distribution. The $Q^2$-statistic is calculated from the portmanteau test of the squared value of the time series. Under the null hypothesis that there is no autocorrelation in the squares of the time series, the $Q^2$ statistic has a $\chi^2$-distribution.

One of the nonlinear models which will be applied to the price changes in the futures markets is the GARCH model. The GARCH model, which assumes that the current variance of the time series is an autoregressive of the past squared values of the time series and of the past variances, can be used to study price volatility in financial markets. The estimation of the GARCH model is obtained using the maximum likelihood algorithm developed by Berndt, Hall, Hall, and Hausman (1974).

The other nonlinear time series model which will be applied to futures prices is the threshold autoregressive model. The major steps for estimating the threshold autoregressive model are identification of the threshold lag, locating the threshold values, and estimating the AR coefficients in each threshold region. To accomplish these, first we need to calculate the TAR-F statistic with different threshold lags and chose the one with the largest test statistic. And then run the arranged recursive regression with this threshold lag, plot t-ratio of the AR coefficient at each recursive step. The places where the t-ratio has large changes are the locations of the threshold values. After obtaining the threshold lag and the threshold values we can estimate the threshold autoregressive model using ordinary least square method.

## 1.5 Overview

This dissertation is organized as follows: after the introduction in Chapter 1, Chapter 2 provides an overview of the analytic foundation to the dissertation. It discusses the financial economic models for price changes in financial markets, the three new test statistics which will be studied and applied to analyze the price changes in futures markets, and the two econometric models which will be used to model the price changes in futures markets. The Monte Carlo experiments for studying the finite sample properties of the three tests will also be discussed in Chapter 2.

Chapter 3 reviews in detail the theories on price movement in financial markets, the efficient market hypothesis, the random walk model, the martingale model, the mean-reverting model, the market anomalies, and the futures markets from which the price movements will be analyzed and modeled in this dissertation.

Chapter 4 discusses the important results from research of nonlinear dynamics which are relevant to the study of price changes in financial markets. The three test statistics, the BDS statistic, the TAR-F statistics, and the $Q^2$ statistics are discussed in the chapter. Two econometric nonlinear time series models, the threshold autoregressive model and the GARCH model will also be discussed.

In Chapter 5, the Monte Carlo method and experiment design are discussed. Chapter 5 also reports the results of the Monte Carlo study of the finite sample properties of the new test statistics. Estimates of empirical power and size of three tests are also discussed.

Chapter 6 details the empirical studies of price changes in futures markets. we will describe the data, analyze the univariate properties of the data, present the test results from the three test statistics, and examine how well can the threshold autoregressive model and the GARCH model fit the price changes in futures markets. Finally, Chapter 7 concludes the dissertation.

# CHAPTER 2

# ANALYTIC FOUNDATION AND OVERVIEW

## 2.1 Introduction

This chapter describes the analytic foundation of the dissertation. In this chapter we briefly discuss theories on financial price determination, economic models of price movements in financial markets, the properties of some statistical tests used in financial economics, the nonlinear econometric time series models for studying financial prices, and Monte Carlo experiments for investigating the finite sample properties of the new test statistics. This chapter serves as an overview. A detailed discussion of these topics is presented in the following three chapters.

The major components of this dissertation are organized and linked as follows. First we will discuss the important theory of financial economics that serves as a principle to econometric model building, the efficient market hypothesis. Then we will review the financial economic models that satisfy the efficient market hypothesis. We will also review the previous methods for testing these models and the empirical results which either support or reject these models. Based on this review, we will discuss the short comings of previous testing methods, and the need for nonlinear methods and models to study price movements in financial markets.

Next we will discuss the need for nonlinear models when modeling complicated time series data as found in financial economics. Then we will review the new statistical tests which are capable of detecting nonlinear serial dependence in time series data. Two econometric nonlinear time series models and the associated estimation methods will also be discussed.

Because the finite sample properties of the new statistical tests are not fully known and because they are important for empirical analyzing of economic and financial data, we use Monte Carlo experiments to investigate them. The results of the Monte Carlo experiments will be presented and discussed in this dissertation.

Finally after investigating the finite sample properties of the tests, we will apply them, and two econometric nonlinear time series models, to futures prices. We will test whether the futures prices have nonlinear serial dependence. After modeling futures prices, we will discuss the estimated models and their relationship to the financial economic theories and models.

## 2.2 Theories and Models in Financial Economics

One important theory in financial economics is the efficient market hypothesis. The efficient market hypothesis requires that market prices reflect all information. In this case it is impossible to build a predictive model based on the information and formulate a profitable trading strategy.

The efficient market hypothesis (EMH) can be divided into three levels if we sort information into three categories.[1] The weak-form EMH asserts that prices fully reflect information contained in historical prices. The semi-strong-form EMH asserts that asset prices reflect not only historical price information but also all publicly known information relevant to the asset. The strong-form EMH asserts that all relevant asset information, including public, private, and historical price information, is fully reflected in asset prices.

In this dissertation, I focus on the weak-form EMH and study the relationship between current price and past prices. Three financial economic models, i.e., the random walk model, the martingale model, and the mean-reverting model are examples of models which relate current price with past prices and which satisfy the weak-form EMH. The following is an overview of these models.

## Random Walk Model

The random walk model of price changes in financial markets states that price changes are random. That is, they are independent across time, and have identical distribution (IID) for all periods with zero mean. It can be written as:

$y_t - y_{t-1} = e_t$ , $e_t$ is IID with zero mean,

[1] Roberts (1967) first made the distinction between three forms of EMH. Later many others, particularly Fama (1970 and 1991), have refined the definitions of three forms of EMH.

where $y_t$ is the price (or the log price) of the asset at time t. When price changes follow the random walk model, the weak-form EMH will be satisfied. Because if we take the expected value of price change, then:

$$E(y_t - y_{t-1}) = E(e_t) = 0 .$$

Therefore the expected price change and the expected return will be zero for the random walk model and, on average, there will be no chance of making a profit from trading in the market.

Martingale Model

The martingale model states that the expected value of price in period t conditioned on the information available at t-1 is the price in period t-1, that is:

$$E(y_t | I_{t-1}) = y_{t-1} ,$$

where $I_{t-1}$ is the information of historical prices at time t-1. The martingale model is less restrictive than the random walk model because it neither requires price changes to be independent nor identically distributed.

Another expression of the martingale model is[2]:

[2] See Leroy (1989).

$$E[(y_t - y_{t-1}) | I_{t-1}] = 0 \, .$$

This shows that the expected price change is zero, so on average it will be impossible to make profit from trading in the market and, consequently, the martingale model also satisfies the weak-form EMH.

Mean-Reverting Model

The mean-reverting model suggests that financial prices follow a random walk over the short-run, but over long-run they maintain a mean-reverting process in which prices revert to their mean (such as the fundamental value). From its definition, we can write the mean-reverting model as follows:

$$y_t = (y_{t-1} + e_t) + f(I_{t-1}) \text{ , } e_t \text{ is IID with zero mean,}$$

where the first term on right hand side is the random walk component, and the second term $f(I_{t-1})$ represents the mean-reverting component.

In the mean-reverting model, financial prices follow random walk over the short-run, so prices are unpredictable. Over a long-run prices are also unpredictable because the "mean" to which prices revert is itself unpredictable.[3] Therefore when prices follow mean-reverting model, on average there will be no chance of marking profit from

[3] See Friedman (1984) and the discussion in Section 3.6 of this dissertation.

trading, and the mean-reverting model satisfies the weak-form EMH.

## Previous Testing Methods and Results of Empirical Study

The empirical test of the weak-form EMH has mainly focused on whether financial prices are predictable based on past prices. Some researchers have investigated the autocorrelation of prices, while others looked predictive behavior of financial-price models. But no conclusive evidence has emerged found in support of or against the weak-form EMH.

Most of the earlier work on the test of the random walk model had relied on the study of the distribution of the returns. The spectral analysis and autocorrelation method were also used to test the random walk model. These tests generally did not reject the random walk model. However, the variance-ratio test on stock returns provides us with evidence against the random walk model.

One test of the martingale model is to examine if past prices can be used to predict future prices and if the expected price change is zero. However, this test is a joint test of the predictive model and the martingale model, it has model specification problem and is difficult to be implemented.

The test of the mean-reverting process is based on the autocorrelation method. Using the autocorrelation test, some researchers have shown the result in support of the mean-reverting process. But later it is argued that the calculated test statistics in these studies are too low and the results are not statistically significant.

The testing methods used in these studies are incapable of detecting nonlinear serial dependence in time series. For testing the weak-form EMH and the martingale model, there is model specification problem for the predictive model. There were also increasing empirical evidence against the models of random walk, martingale, and mean-reverting. If financial prices follow a nonlinear process, the previous linear methods will be deficient for analyzing financial prices. Therefore we need to seek new techniques and models for studying financial prices.

## 2.3 New Test Statistics

The three newly developed test statistics, the BDS statistic, the TAR-F statistic, and the $Q^2$ statistic, are studied and then applied to futures prices in this dissertation. These three test statistics are particularly useful in financial economics because they have the ability to detect nonlinear serial dependence in time series where the linear method of autocorrelation sometimes fail. They can also help us to identify the nature of serial dependence and to model financial prices.

### The BDS Statistic

The BDS statistic is developed from the correlation integral for detecting time series which is not independently and identically distributed (IID). The BDS statistic is defined by:

$$W_{M,T}(\epsilon) = T^{1/2} (C_{M,T}(\epsilon) - C_{1,T}(\epsilon)^M)/\sigma_{M,T}(\epsilon) ,$$

where T is the sample size, $\epsilon$ is the correlation length, M is the embedding dimension, $C_{M,T}(\epsilon)$ is the correlation integral, and $\sigma_{M,T}(\epsilon)$ is the estimate of the standard error under the IID null hypothesis. And the correlation integral is given by:

$$C_{M,T}(\epsilon)=\frac{2}{(T-M+1)(T-M)}\sum_{1\le i<j\le T} I_\epsilon(x(M)_i\ ,\ x(M)_j)$$

where indicator function $I_\epsilon(.)$ is defined by:

$$I_\epsilon(x(M)_i,x(M)_j) = 1 \quad \text{if sup norm } \|x(M)_i\text{-}x(M)_j\| < \epsilon ,$$

$$I_\epsilon(x(M)_i,x(M)_j) = 0 \quad \text{elsewhere} \quad .$$

Under the IID null hypothesis, the BDS statistic will be asymptotically standard normal. Under the alternative hypothesis, the BDS statistic will not have standard normal distribution and will have large frequency of being rejected when the alternative time series departs from the IID null hypothesis to a certain degree.

The BDS statistic can be applied to time series to detect the serial dependence in time series. It also can be applied to the linear filtered data to detect nonlinearity in time series. Another use of the BDS statistic is to test the model adequacy by checking whether the residuals of the model are IID.

<u>The TAR-F Statistic</u>

The TAR-F statistic is designed to detect time series of the threshold autoregressive process. The test statistic is given by:

$$F(p,d) = \frac{(\sum e_t^2 - \sum \epsilon_t^2)/(p+1)}{\sum e_t^2/(T-d-b-p-h)}$$

where p and d are respectively the AR order and the threshold lag in the test, $e_t$ are the standardized predictive residuals from the arranged AR recursive regression of the time series, and $\epsilon_t$ are the residuals of the regression of $e_t$ on the lagged values of the time series, b is the starting point of the arranged recursive regression, h=max(1+p-d, 0).

Under the null hypothesis that the time series is linear, the TAR-F statistic has an F-distribution with p+1 and T-d-b-p-h degrees of freedom. If the time series has model changes in different regions of the threshold variable, then the TAR-F statistic will not have an F-distribution and will fall into the rejection range with large frequency if the alternative time series is sufficiently apart from the linear hypothesis.

<u>The $Q^2$-statistic</u>

The $Q^2$-statistic is developed for identifying the nonlinear time series with autocorrelation in squared values of time series. The $Q^2$-statistic is defined by the following:

$$Q^2(p)=T(T+2)\Sigma_{k=1,p}\, r^2(k)/(T-k)\ ,$$

where r(k) is the autocorrelation function of the squared values of the time series.

The null hypothesis of the $Q^2$-statistic is that the time series has no autocorrelation in its squared values. Under the null hypothesis, the $Q^2$-statistic is distributed $\chi^2(p)$. When the time series has autocorrelation in its squared values, then the $Q^2$-statistic will not have the $\chi^2$-distribution and will fall into the rejection region with large frequency when the autocorrelation in the squared values of the time series is not too small. The $Q^2$-statistic is an ideal technique for detecting ARCH type time series because the method of autocorrelation generally fails.

## Finite Sample Properties of New Test Statistics

The three statistical tests reviewed in this section have ability to detect nonlinear serial dependence. However, we do not have full information on the finite sample properties of the tests. Past studies on these tests have only shown the results of the tests at few sample sizes, usually less than three sample sizes, under the null hypotheses of the tests. Under the alternative hypotheses of the tests, the results of the tests were known only at few sample sizes for several types of time series with one or two parameter values of the time series, where the parameter values of the time series represent the departure from the null hypotheses of the tests.

Clearly, our knowledge on the finite sample properties of these tests so far are very limited. The finite sample properties of these tests are important for empirical study of economic and financial data where the sample size is always finite. Thus we need to know the results of the tests at more sample sizes, for more types of time series, and with more parameter values of the time series. The Monte Carlo method discussed in the next section is a effective tool for study the finite sample properties of these tests.

## 2.4 Monte Carlo Experiment

The three new statistical tests discussed in last section are capable of detecting nonlinear serial dependence in time series data, and they can help us model financial price movements. But the behavior of the tests with finite sample is not fully understood. Because the finite sample properties of these tests can not be derived analytically, the investigation of the finite sample properties of these tests has to rely on the method of Monte Carlo experiment.

Monte Carlo experiment is a procedure in which quantities of interest are studied based on generated sample data. A Monte Carlo study usually has many experiments. Each experiment will have: a) the quantities of interest, which in this dissertation are the test statistics; b) a data generating process (DGP); and c) some number of replications, where each replication involves generating a single set of data from DGP and calculating the quantities of interest. Usually a set of related experiments are conducted in which the sample size and other aspects of the DGP (such as parameter values) are varied in order to see how such variations affect the quantities of interest.

When the number of experiments is small, the results of Monte Carlo experiments can be presented in tabular form. When the number of experiments is large, as in this dissertation, the results of Monte Carlo experiments are difficult to present in tabular form. Therefore we use response surfaces to statistically summarize Monte Carlo experiments results, where the quantities of interest are related to the sample size and to other aspects of the DGP that vary across the experiments.

Generating random numbers is an important part of the Monte Carlo experiment. The random numbers with uniform distribution U(0,1) are generated first and the random numbers with other distributions can be generated by using random numbers of U(0,1). In this dissertation, the Multiplicative Congruential Generator and the shuffling method are used to generate random numbers of U(0,1), and the Box-Muller bivariate method is used to generate random numbers with normal N(0,1) distribution. Finally, other DGPs can be derived from these random numbers.

The following are the data generating processes (DGPs) used in our Monte Carlo experiments for study the finite sample properties of three tests, where $x_t$ is generated sample data for Monte Carlo experiments, and $e_t \sim N(0,1)$ is the error term:

**1) DGP of IID Time Series**

1.1) Standard normal distribution: $x_t \sim N(0,1)$.

1.2) Uniform distribution: $x_t \sim U(0,1)$.

1.3) Bimodal mixture of normals: $x_t \sim \{0.5\ N(0,1) + 0.5\ N(\alpha,\beta^2)\}$.

**2) DGP of Linear Time Series**

2.1) Linear autoregressive of order 1: $x_t = \alpha x_{t-1} + e_t$ .

2.2) Linear moving average of order 1: $x_t = \alpha e_{t-1} + e_t$ .

**3) DGP of Nonlinear Time Series**

3.1) Nonlinear autoregressive of order 1: $x_t = \alpha x_{t-1}(1-x_{t-1})/(1+x_{t-1}^2) + e_t$ .

3.2) Nonlinear moving average of order 1: $x_t = \alpha e_{t-1} e_{t-2} + e_t$ .

3.3) Threshold autoregressive (TAR) process:

$$x_t = \alpha x_{t-1} + e_t \ , \ \text{if } x_{t-r} \leq 0 \ ,$$

$$x_t = - \alpha x_{t-1} + e_t \ , \ \text{if } x_{t-r} > 0 \ .$$

3.4) GARCH process:

$$x_t = h_t^{1/2} e_t \ ,$$

$$h_t = 1 + \alpha x^2_{t-1} + \beta h_{t-1} \ .$$

In the Monte Carlo study of the test statistics, we use four sample sizes and several parameter values of DGP for each type of DGP. Then we construct the response surfaces of the test statistics showing the effect of sample size and the parameter of the DGP on the test statistics. A full discussion of Monte Carlo experimental design is found in Chapter 5.

## 2.5 Econometric Nonlinear Time-Series Models

The two nonlinear econometric time-series models which are useful for modeling financial prices are the threshold autoregressive (TAR) model and the generalized autoregressive conditional heteroskedasticity (GARCH) model. The TAR model has nonlinearity in conditional mean, and the GARCH model has nonlinearity in conditional variance. These two models are employed for modeling futures prices in this dissertation. In the Monte Carlo study of the new statistical tests, we also use the special cases of these tow model in the data generating processes.

The TAR model is defined by:

$$x_t = \alpha^{(j)}{}_0 + \sum_{i=1,p} \alpha^{(j)}{}_i x_{t-i} + \epsilon^{(j)}{}_t \,, \quad r_{j-1} \leq x_{t-d} < r_j \,,$$

where $j=1,\ldots,k+1$, k is the number of threshold regions, p is the AR order, d is the threshold lag, $x_{t-d}$ is the threshold variable. The threshold values of the model are $\infty = r_0 < r_1 < \ldots < r_k < r_{k+1} = \infty$; for each j, $\{\epsilon^{(j)}{}_t\}$ is IID with zero mean and variance $\sigma_j^2$. This model is a piece-wise linear AR model, since it follows a different linear AR processes when the threshold variable $x_{t-d}$ falls into a different threshold region. The overall model is not linear when there are at least two regions with two linear processes.

In Chapter 4 and Chapter 6 we will see that in order to estimate the TAR model, we first have to select a sufficiently large AR order p, and then calculate the TAR-F statistic for a range of threshold lags. The threshold lags at which the TAR-F statistic has large values can be used as the possible threshold lag. At the selected threshold lag d, we implement an arranged recursive regression, and plot the t-ratios of the AR coefficients. The values of the threshold variable at which the t-ratios have large changes can be the possible threshold values. Once the threshold lag and the threshold values are identified, we can estimate the TAR model using ordinary least squares. More details are found in Chapter 4 and Chapter 6.

If financial price changes follow the TAR model, then they are not IID. Furthermore, the conditional mean of the price changes is not constant. Therefore the TAR model is not consistent with the random walk model, the martingale model, and the mean-reverting model. As to whether the TAR model violates the weak-form EMH, we have to examine whether the TAR model can be used to make profitable trading strategy.

The second model used in Chapter 6 is the GARCH model. It is defined as:

$$x_t \sim N(0,h_t)\ ,$$

$$h_t = \alpha_0 + \sum_{i=1,p} \alpha_i x_{t-i}^2 + \sum_{j=1,q} \beta_j h_{t-j}\ .$$

In this model, the conditional mean of $x_t$ is zero, and the conditional variance is $h_t$. The conditional variance is an autoregressive process of the past values of the time series and of the past variances.

The estimation of the GARCH type is obtained by using the maximum likelihood method. For the GARCH model presented here, its log likelihood function is[4]:

$$L_T(\theta) = T^{-1}\sum_{t=1,T} l_t(\theta) ,$$

$$l_t(\theta) = -(1/2) \log h_t - (1/2) x_t^2 h_t^{-1} ,$$

where T is the sample size. The Berndt, Hall, Hall and Hausman (1974) algorithm will be used to obtain the maximum likelihood estimates.

The simple GARCH model discussed here has zero conditional mean. In a more general GARCH models we can have non-zero conditional mean, but the estimation of the model is similar to the estimation of the simple GARCH model.

When price changes follow the GARCH model, they are not IID. In this case the price changes violate the random walk model and the mean-reverting model. If the GARCH model of the price changes has autoregressive conditional mean, the price changes also violate the martingale model. In the later case, we have to investigate if the GARCH model can be used to make profitable trading strategy before we conclude whether the prices obey or violate the weak-form EMH.

## 2.6 Summary

One important theory in financial economics is the efficient market hypothesis

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[4] Apart from some constant, see Bollerslev (1986).

(EMH). The weak-form EMH asserts that financial prices reflect information of historical prices. Thus it is impossible to use historical prices to formulate a profitable trading strategy.

The random walk model, the martingale model, and the mean-reverting model are examples of models that satisfy the weak-form EMH. If financial prices follow these models, then on average it will be impossible to make a profit from trading. These models, to a certain extent, helped us understand price movements in financial markets. However, there are empirical studies which reject these models.

Earlier methods in the study of price movements in financial markets were limited to the techniques of autocorrelation, calculation of distribution, and linear models. These methods can fail to detect nonlinear serial dependence in financial prices. If there is a nonlinear component in financial prices, these methods can not help us analyze and model the nonlinearity in financial markets either.

The three new test statistics, the BDS statistic, the TAR-F statistic, and the $Q^2$ statistic, has the ability to detect nonlinear serial dependence in time series where the linear method sometimes fail. These test statistics can also help us identify and model serial dependence in time series, which is very important for studying price movements in financial markets.

For these three test statistics, however, their finite sample properties are not fully known. In this dissertation we use Monte Carlo simulation to study the finite sample properties of these three test statistics. In the Monte Carlo experiment, we consider several data generating processes (DGPs), many parameter values of the DGP, and four sample sizes. Finally we use the response surface to present the results of Monte Carlo experiments.

Two nonlinear econometric time series models that are useful for studying price movements in financial markets are the TAR model and the GARCH model. They are applied in this dissertation for study of futures prices. The TAR model is an example of nonlinear time series model which has nonlinearity in conditional mean. The TAR model is inconsistent with the random walk model, the martingale model, and the mean-reverting model. The GARCH model has conditional variance which can be useful in studying price volatility in financial markets. The GARCH model is inconsistent with the random walk model and the mean-reverting model. When GARCH has autoregressive conditional mean, it is also inconsistent with the martingale model. To determine whether the TAR model and the GARCH model violate the weak-form EMH, we have to investigate if the conditional mean of these models can be used to formulate profitable trading strategy.

# CHAPTER 3

# FINANCIAL ECONOMICS AND FUTURES MARKETS

## 3.1 Introduction

Financial markets play an important role in economics. The movements of financial price are always concerned by investors and researchers. Daily news reports on television, radio, and newspaper inform us, for example, of latest stock market index values, currency exchange rates, and gold prices. For most of time, financial prices do not change very much. But some times financial prices have substantial changes. For example, the market break of stock market on October 19, 1987 shocked the nation and the world, and the volatile oil price changes during the gulf war arose concerns of industrial countries. These sudden changes of financial prices will not only affect investors' wealth, but they will also have short term and long term impact on industries and economy.

For investors and scholars, it is often desirable to monitor price behavior in financial market frequently and try to understand the probable development of financial prices in the future. The current and future interest rates will affect people's decision on purchasing a house and financing. The changes of commodity prices will alter producer's decisions on output, manufacture's decision on production procedure, and consumer's consumption pattern. The changes of stock prices will affect an investor's decision to adjust the composition of investment portfolios. While people are planning vacation abroad, their decision of whether to buy the foreign currency now or buy it at the destination will be depended upon the behavior of currency exchange rates.

Financial market prices are related to daily economic activities. And the financial market prices are constantly changing. A fundamental hypothesis about price movements in financial markets is the efficient market hypothesis (EMH) which states that the prices of an asset fully reflect all information. When the market is efficient, prices fully reflect all information and it will be impossible to make excessive profits in the market base on the information.

The early studies on security price were that of Williams (1938) and Graham and Dodd (1934). They stated that securities have intrinsic fundamental values which equal to the discounted cash flows of securities. The determination of fundamental values of a security involves analyzing demand for the products, possible future development of substitutes, the probability of recession, changes in regulatory environment, and other information relevant to future profitability of the firm associated with the security. The method of analyzing financial prices base on these concepts is called fundamental analysis. Unfortunately, evidence have shown that the fundamental analysis does not work well in predicting financial asset prices. Thus other models were developed to study financial prices.

The random walk model, developed at the turn of the century, did not get much attention of economists until 1950s and 1960s. A basic version of the random walk model assumes that the time series of an asset's price changes is independent and identically distributed with a normal distribution. Empirical studies have suggested that distributions other than the normal distribution may be more appropriate for the random walk model, and some studies even rejected random walk model of price movements in financial markets.

The martingale process and mean-reverting process are the two more recent models describing price movements in financial markets. The martingale process permits the price changes to be dependent and distributed differently, but requires the expected price change to be zero. In a mean-reverting process, the price changes are the combined results of a random walk and a process which reverts to the mean of the asset price (or the fundamental value of the asset).

In this chapter, I briefly review the basic concepts and the properties of futures markets, on which the empirical study of the dissertation will be based. Then I discuss the efficient market hypothesis, the random walk model, the martingale model, and the mean-reverting model. The recent findings on market anomalies in financial markets are also discussed. A summary section ends the chapter.

## 3.2 Futures and Futures Markets[1]

Traditionally, futures markets have been recognized as meeting the needs of three

[1] A good literature on this subject is the book by Kolb (1984).

groups of futures market users: those who wish to discover commodity prices in the future; those who wish to speculate; and those who wish to transfer unwanted risk to some other party. Kamara (1989) also pointed out that futures markets served as a clearing house for information in which informed traders' information is transmitted to uninformed traders. Futures trading also can stabilize the cash market and to improve the inter-temporal allocation of resources because the traders in the cash markets have the options to store the commodity and trade on futures markets.

In buying or selling a futures contract a trader agrees to receive or deliver a given commodity at a certain time in the future for a price that is determined now. In such a circumstance, it is not surprising that there is some relationship between futures price and the price that people expect to prevail for the commodity at the delivery date specified in the futures contract. While the exact nature of that relationship is unclear, the relationship is predictable to a high degree. By using the information contained in the futures price today, it is possible to form estimates of what the price of a given commodity will be at a certain time in the future. The forecast of future price that can be drawn from the futures market compare in accuracy quite favorably with other types of forecasts. Futures markets serve a social purpose by helping people to form a better idea of what the future prices will be, so that they can make their consumption and investment decisions more wisely based on the best available information.

Without doubt, however, futures markets also provide the opportunity for speculation. Speculators who believe that the price will move away from the current price can profit by buying or selling the futures today and make a reverse trade later. And one trader's profit is made from other traders' losses. But even if one regards speculative activity as evil or immoral, there is strong evidence that the presence of speculators benefits the other users of futures markets by helping to provide liquidity in the futures markets.

Many futures markets participants trade futures in order to avoid some unwanted risk. A classical example is that the farmer who trades in futures market to avoid risk associated the uncertain price at harvest of the crop he or she is producing. The activity of hedging is the prime social rationale for futures trading. By being able to transfer risk to other parties via the futures market, economic activity in general is enhanced.

Some people argue that futures prices equal expected future spot prices. For example, if the futures price of wheat that will be delivered in six months is $5.54 per bushel, then, according to this argument, the market expects the price of wheat to be $5.54 six months from now. The expectation is the product of all participants in the market, who vote on the correct future spot price by their trading on the price of the commodity at the delivery date in the future. If a trader foresees correctly that the prevailing futures price of a good delivered in six months exceeds what he expects that good to be worth, then he can profit by selling futures contract on that good.

There are three reasons that the equality of futures price and spot price at the time of delivery might not hold. First, the risk-bearing services of speculator will be forthcoming only if the futures price differs from the expected future spot price. Second, the feature of daily resettlement prevalent in futures markets could cause futures prices and expected future spot prices to diverge under certain circumstance. Third, there is an alternative concept of futures pricing that relies on carrying charges to determine futures pricing relationship.

Futures markets participants are divided into two groups: hedgers and speculators. Hedgers enter the futures markets to reduce a pre-existing risk, while speculator trade in the hope of profit. Speculators in the futures markets bear certain risks. Risk-averse speculators will trade in the market only if the expected profit is large enough to compensate for risk exposure. It is generally believed that most participants in financial markets are risk averse.

The trader who buys a contract is said to have a long position. The seller of a contract is said to have a short position. A speculator will take a long position in the futures markets only if the expected future spot price of the commodity is greater than the current futures price. Otherwise, the speculator must not expect to make any profit. On the other hand, a hedger who wants to avoid unwanted risk need to take a short position. The hedger must be willing to sell the futures contract at a price that is less than the expected future spot price of the commodity. Otherwise, the hedger cannot induce the speculator to accept long side of contract. For the same reason, if the hedger needs to be long to reduce his risk, the speculator has to be short. Then the hedger has to buy the futures contract at the price higher then the expected future spot price. In this case, the futures price will be higher than the expected future spot price. From this point of view, the hedger is, in effect, buying insurance from the speculator. The hedger transfers his unwanted risk to the speculator, and pays the speculator for bearing the risk. The payment to the speculator is the difference between the futures price and the expected future spot price. Even so, the speculator does not receive any sure payment. The speculator must still wait for the expected future spot price to materialize to capture the profit expected for bearing the risk.

Hedgers are not confined to the hedging activities, they may also participate in speculation. Kamara (1989) argued that the optimal hedge is generally a mixture of hedging and speculation. A producer will not only use the futures markets to hedge on price uncertainty, but will also use the futures price to make production decision which involves speculating on futures prices and taking a certain futures position. Therefore both hedgers and speculators speculate. While speculators speculate on futures price level alone (trading followed by reverse trading in hope of profiting from intervening price changes), the hedgers speculate on both futures price level and the price difference between futures price and spot price.

The types of futures contracts that are traded fall into five categories. The underlying good traded may be an agricultural or metallurgical commodity, an interest-earning asset, a foreign currency, or a stock index. Contracts for more than forty different goods are currently available.

For agricultural commodity futures, contracts are traded in grains (corn, oats, and wheat), oil and meal (soybeans, soymeal, and soyoil, and sunflower seed and oil), livestock (live hogs and cattle and pork bellies), poultry (eggs and live broilers), forest products (lumber and plywood), textiles (cotton), and foodstuffs (cocoa, coffee, orange juice, potatoes, and sugar). For many of these commodities, several different contracts are available for different grades or types of commodity in question. For most of the good, there are also different contracts with different delivery times.

The metallurgical futures includes the genuine metals, as well as petroleum contracts. Metals and petroleum are treated in a similar way because both of them share a important common characteristic: they can be stored indefinitely. Among the metals, contracts are traded on gold, silver, silver coins, platinum, palladium, and copper. Of the petroleum products, heating oil, crude oil, gasoline, and propane are traded on futures markets.

The futures trading on interest-bearing assets started only in 1975, but the growth of this market has been tremendous. Today contracts are traded on Treasury bills, notes, and bonds, on bank Certificates of Deposit, Eurodollar deposits, and GNMAs, which are government backed single-family mortgages.

Active trading of foreign currency futures dates back to the inception of freely floating exchange rates in the early 1970s. Contracts are traded in the British pound, the Canadian dollar, the Japanese yen, the Swiss franc, and the West German mark. Contracts are also listed on French francs, Dutch guilders, and Mexico peso, but these have met with only limited success. The foreign exchange futures market represents one case of a futures market existing in the face of a truly active forward market, which is many times larger than the futures market. Many people believe that the present of the forward market deterred the introduction of foreign exchange futures.

The last group of futures contracts is for stock indices. Beginning only in 1982, these contracts has been quite successful, with trading on four broad market indices. Four different exchanges trade contracts on three different indices: the Standard and Poor's

500, the Major Market Index, the New York Stock Exchange Index, and the Value Line Index. In addition, numerous contracts on industry indices are now trading as well. The stock index contracts do not admit the actual delivery of the underlying asset. A trader's obligation must be fulfilled by a reversing trade or a cash settlement at the time of maturity.

## 3.3 Financial Price Movement and Efficient Market Hypothesis

Prices of financial assets, such as the prices treasury bills, stocks, and futures, are determined in the markets, just as the price of any good or service. The prices of treasury bills and bonds reflect the rate of interest in the economy. The price of a stock reflects the expected future returns from holding the stock. The price of a commodity futures reflects the expected supply and demand of the commodity down the time path. The prices of futures, including financial futures and currency futures, also reflect fees people are willing to pay to avoid future transaction uncertainty.

Financial asset prices are influenced by people's opinions. Different people have different opinions about the future path of economic activities and the future path of asset prices. These opinions will be translated into selling and buying in the market. Transactions involving financial assets can be completed quickly in the markets. The two defining characteristics of financial assets, opinion-influenced prices and short transaction time, cause financial price changes to be frequent and difficult to forecast. As time goes by, new information (economic or non-economic) continuously emerges and affects people's opinions about the future path of economic activities. People will adjust their holdings of financial assets accordingly. As a result, financial prices will constantly change when the markets are open for trading.

Then, how will financial asset prices change over the time? Theoretically, if current prices provide arbitrage profit opportunities, then these opportunities will be pursued until price changes cause then to disappear. Suppose it is a certainty that the price of an asset will drop significantly tomorrow. Selling short today (borrowing the asset today while agreeing to repay some time later) and buying back tomorrow would yield a profit. Therefore the price of the asset will drop instantly, making this scheme unprofitable. The opposite process is set in motion if the price of an asset is certain to rise tomorrow.

In theory, financial asset prices will move in certain course and not allow profit from selling and buying over time. It will not allow profit from changing positions across different assets either. If one asset provides higher expected returns than others, then its price will be higher, its return derived from price change later will be reduced until its expected return equal the expected return of the others. Of course, when we calculate the expected returns we also have to deduct the risk premiums. Some riskier assets may have higher nominal expected returns, but after deducting the risk premium their expected returns will be equal to the expected returns from other assets.

As mentioned in Chapter 1, a fundamental hypothesis regarding the way prices change in financial markets is the efficient market hypothesis (EMH). A market is efficient if the prices in the market fully reflect all information contained in a given information set. The information set can include historical prices, trading volume, publicly known market information, economic information, political information, insider corporate information, insider government information, etc.. Roberts (1967) proposed that three forms of market efficiency can be defined according to the domain of the information set: the weak-form, the semi-strong form, and the strong form efficient market hypothesis.

The weak-form efficient market hypothesis claims that prices in a market fully reflect all information contained in past prices. The semi-strong form EMH claims that market prices fully reflect all publicly available information, including past volume and price data. The strong form EMH states that market prices reflect all information, whether public or private. Private information includes information possessed only by corporate insiders and government officials[2]. If the market is efficient, then given the information set, we can not expect to make profit in the market, because the prices fully reflect the information.

Fama (1970 and 1991) provided interpretations and tests for the three types of market efficient hypothesis, these are: the **weak-form test**, the test for return predictability, testing how well do past returns predict future returns;[3] the **semi-strong-form test**, the event test, testing how quickly do security prices reflect public information

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[2] See Kolb (1985), Fama (1970, 1991).

[3] Fama (1991) also suggested to include other variable such as dividend yield and earnings price ratios in testing of weak form EMH. But others, such as Ross (1989) and Malkiel (1989), suggested to use just past prices in testing of weak form EMH. In this dissertation I take the later position.

announcements such as dividend, discount rate, or government subsidy; and the **strong-form test**, test for private information, testing whether any investors have private information that is not fully reflected in the market prices. The weak-form EMH is the least restrictive form of the EMH. If the strong form or semi-strong form EMH is valid, the weak-form EMH must be valid also. This dissertation is focused on the weak-form EMH, and will study if financial price is predictable from past prices.

Three financial economic models that satisfy the weak-form EMH are the martingale model, the random walk model, and the mean-reverting model. The martingale model requires the expected price change to be zero. Therefore the expected return is zero, and the martingale model satisfies the weak-form EMH. The random walk model imposes even stricter conditions. It requires not only that the expected value of returns to be zero, but that the returns to be independent and identical distributed.

To test the hypothesis of weak-form market efficiency, a common procedure is to investigate the differences between the predicted prices based on historical prices and the current prices of the asset. If the time series of price differences has an expected value of zero, then prices incorporate all the past information and do not allow excessive gain on average. We can see that this test is a joint test of the asset price formation model and of the efficient market hypothesis. If the predictive model is not correctly specified, the test may lead to incorrect acceptance or rejection of the efficient market hypothesis. Therefore linear predictive models should not be used for testing the efficient market hypothesis if there is nonlinearity in the time series of financial prices[4]. While

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[4] See Fama (1976), and Heimstra (1990).

many studies indicated linear models do not work for price changes in financial markets,[5] some recent studies have found the presence of nonlinearities in financial prices[6].

Even if we set aside the question of jointness in the tests of market efficiency, the efficient market hypothesis could be violated under certain economic conditions. Danthine (1977) pointed out several possibilities: a) the only solution for prices is a corner solution - when rational agents are constrained by the availability of goods to exploit excess (abnormal) profit and; b) decreasing returns on technology will cause non-zero expected price differentials. In addition Danthine stated that the traditional study of market efficiency by testing for zero autocorrelation in returns is actually the simultaneous test of market efficiency, perfect competition, risk neutrality, constant returns to scale, and the impossibility of corner solution. So zero autocorrelation in returns is, for this reason, not a proper test of efficiency in the commodity market. Although Danthine's conclusions were based on the commodity cash markets, they extend to other financial markets as well.

The efficient market hypothesis also has been attacked by arguments that certain trading rules are effective in generating profits. Taylor (1983) presented some evidence that trading rules constructed from price-trend models are profitable when applied to six

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[5] The studies of Working (1934), Cowles and Jones (1937), and Kendall (1953). found the serial correlation was essentially zero for price changes in financial markets, which basically rejected the linear time series model.

[6] See, for instant, Hinich and Patterson (1989), Tsay (1989), Scheinkman and LeBaron (1989), Hsieh (1989, 1990).

futures contracts. This shows that it is possible to use past prices as inputs in trading rules and obtain profit in financial markets. This is contrary to the efficient market hypothesis.

In the following sections I review several models in financial economics, their relation to the efficient market hypothesis, and some empirical tests of these models.

## 3.4 Random Walk Model

The random walk model of price movements in financial markets was first developed by Bachelier (1900) at the beginning of the century, and modified later by many others. Bachelier, in his original doctoral dissertation paper, built a random walk model for financial price movements based on the following assumptions: a) the price change is stochastic and has a probability distribution; b) the price changes are independent, i.e., the price change in one period will not depend on or affect the price change in another period; c) the time series of the price changes of a particular asset will have an identical probability distribution at each point in time. These three assumptions comprise what is known today as the IID property of the random walk model.

From the above assumptions, how can we know the probability distribution of price changes in financial markets? Suppose $P_{x,t1}dx$ is the probability that the price changes by x during the time span of $t_1$, and $P_{z,t1+t2}dz$ is the probability that the price changes by z during the time span of $t_1+t_2$. Then from the IID property of the random walk, the probability distribution function of the price changes must satisfy the following integral equation:

$$P_{z,t_1+t_2} = \int_{-\infty}^{+\infty} P_{z-x,t_2} P_{x,t_1}\, dx \qquad (3.4.1)$$

which is a continuous stochastic process, like the Brownian motion in heat diffusion process. Equation (3.4.1) states that the probability that the price changes by z during time $t_1+t_2$ is the product of probability that the price changes by z-x during time $t_1$ and probability that it changes by x during time $t_2$, summing over all possible x. In solving this integral equation, Bachelier gave the following distribution function as the solution:

$$P_{x,t} = \frac{1}{2\pi k\, t^{1/2}} \exp\left(-\frac{x^2}{4\pi k^2 t}\right) \qquad (3.4.2)$$

where k is a constant related to the variance of the distribution. This equation gives us the probability of the price changes by x amount during time interval of t. This solution indeed satisfies the integral equation (3.4.1). It is not difficult to see that this is a normal (Gaussian) distribution with mean of zero and variance $4\pi k^2 t$. Bachelier calculated the distribution of price changes of French government bond and compared it to the normal distribution, and obtained what was considered a very impressive result for that time.

Working (1958) developed a theory of anticipatory price in financial markets which justified the random walk model. In the theory, traders are assumed to seek information to guide their price formation. A trader can look at many aspects in the economy for the information. There are many kinds of information that influence financial prices. The information flow is continuous through both public and private channels, and with the influence of information the price changes frequently. The information itself is unpredictable, so the price change is also unpredictable. From this model it is obvious that the price changes in a given period in financial markets depend on the information arrives during the trading period and the information flow rate.

In the random walk model, financial prices are patternless, unpredictable, not smooth nor deterministic over time. This contradicts the analysis of fundamental values and implies that financial prices are exempt from the laws of supply and demand. Roberts (1959) pointed out that the assumption that financial prices adjust instantaneously to new information, and that financial prices are unpredictable, is what we should expect from the efficient market hypothesis and what the random walk model implies. On the other hand, if financial prices adjust slowly to the new information and move in a predictable manner, then profitable trading opportunities that are not being exploited would exist. This would contradict to the efficient market hypothesis.

Bachelier's solution of random walk model, however, has two problems. First, the solution in equation (3.4.2) is not the only solution to equation (3.4.1). Cootner (1964) pointed out that there are many probability density functions that satisfy equation (3.4.1). The binomial, Poisson, geometrical, and compound Poisson distributions, for instance, all satisfy equation (3.4.1). For Bachelier's solution to hold exclusively, the density function must: a) be differentiable with respect to time t; b) have first and second partial derivatives with respect to price change and; c) have finite mean and variance. These assumptions will be examined later. The second problem with Bachelier's solution is that it allows for negative prices. In reality the price of an asset can not fall below zero. So Osborne (1959), Samuelson (1955), and Alexander (1961) proposed using log-price changes (or returns) rather than actual level of price changes in the probability distribution function. Osborne investigated closing stock prices from the New York Stock Exchange and found that the distribution of price changes is not normal. Osborne obtained a better fit of stock prices by using log price changes. He argued that investors look at asset return rather than the price level, and stated that a change in the asset price from $10 to $11 has the same psychological effect as a change from $100 to $110, because the returns are same in both cases. This justifies the "log-normal" distribution of price changes in financial markets.

Early work by Kendall (1953) showed that stock prices appear to follow a random walk. Alexander (1961) studied stock prices in various periods and concluded that stock price changes also appear to be random over time. But Alexander also noticed that a price move, once initiated, tends to persist. In particular, if the stock market has moved up x-percent it is likely to move up more than x-percent further before it moves down by x-percent. This indicates the price changes in the stock market may not be independent from each other.

Lo and Mackinlay (1988) used a variance-ratio statistic to test the random walk hypothesis. The idea of their test is that, if the returns of an asset follow the random walk and Gaussian distribution, even with heteroskedasticity, the variance of the monthly returns should be four times the variance of the weekly returns. The results of variance-ratio test on various stock returns led to a rejection of the random walk hypothesis. Poterba and Summers (1988) also calculated ratio of variance of K-period return divided by K over variance of one-period return and found the ratio decrease with the increase of K. And they rejected the random walk model as well.

Random Walk with Stable Paretian Distribution

Compared to the normal distribution, the distributions of log-price changes of empirical data show the thick tails (i.e., more observations at both lower end and higher end of the distribution compared to the normal distribution). Concerned with the thick tails of the distribution of the log-price changes in financial market, Mandelbrot (1963) felt that thick tails can be captured by a stable Paretian distribution. Unlike the normal distribution, the stable Paretian distribution can have infinite variance. The following is the logarithm of the characteristic function[7] of the stable Paretian distribution:

$$\log\varphi(t) = i\delta t-\gamma|t|^{\alpha}[1+i\beta(t/|t|)\tan(\alpha\pi/2)] \qquad (3.4.3)$$

where i is the square root of -1. The four parameters $\alpha$, $\beta$, $\delta$, and $\gamma$ determine the distribution function. Specifically, $\alpha$ measures the tails of the distribution, $\delta$ is the location of the distribution, $\gamma$ is the scale parameter, and $\beta$ measures the skewness of the distribution. When $1<\alpha<2$, the mean of the distribution exists and variance of the distribution is infinite. For a normal distribution, $\alpha=2$. If $\beta=0$, then the distribution is symmetric.

[7] The characteristic function is another way to represent random variable. It is defined by $\varphi(t) = E(e^{itx})$, where x is the random variable.

The stable Paretian distribution has two important properties: 1) the distribution is invariant under addition, that is, the sum of random variables which have stable Paretian distribution with same $\alpha$ and $\beta$ is a random variable which has stable Paretian distribution with the same $\alpha$ and $\beta$, and; 2) the stable Paretian distribution is the only possible limiting distribution for independent identical distributed random variables[8].

Mandelbrot (1963) studied cotton prices. He computed the sample second moments from daily first-differences of the logs of cotton prices for increasing sample of from 1 to 1300 observations. As the sample size is increased, the sample moment does not settle down to any limiting value but continues to vary in an erratic fashion. The sample moment behaved as the stable Paretian hypothesis predicted. The estimate of the $\alpha$ parameter in the stable Paretian distribution revealed that the value of $\alpha$ appeared to be 1.7 for the cotton prices being studied, where for normal distribution the value of $\alpha$ should be equal 2. This shows that the log-price changes of cotton are better described by stable Paretian distribution rather than normal distribution.

Fama (1965) analyzed daily log-price changes of thirty stocks in the Dow-Jones Industrial Average indices. In every case the empirical distribution was thick-tailed, contrary to the assumption of a normal distribution. Estimates of $\alpha$ were consistently less than 2. Fama concluded that the stable Paretian hypothesis is better than the Gaussian hypothesis in describing the empirical data. J. C. So (1987) analyzed currency futures, and found that $\alpha$ is less than 2. He concluded that most currency futures price changes

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[8] With normal distribution being a special case of stable Paretian distribution, which also has this property. See Fama (1965).

being studied are adequately explained by stable Paretian distributions.

## Random Walk with Sub Gaussian Distribution

From anticipatory price theory of Working (1951) and the discussion in earlier part of this section, we can see that the price change in a given period, say one day, will depend on the information arrives in the period. If there are more information arrive during the day, then the assets will be traded many times, and the price will likely to have large change. This prompted Clark (1973) to use a class of distributions subordinate to the Gaussian distribution (Clark called it the sub Gaussian distribution) to describe the daily price change in futures market.

Clark (1973) studied daily cotton futures prices during 1945-1955, and also noted that distribution of daily price changes does not fit into a Gaussian distribution. Clark argued that daily price changes are the results of many independent events. The futures price evolves at different rates during identical time intervals, depending on the arrival rate of market information. Clark concluded that daily price changes are normally distributed when measured from transaction to transaction, not measured over fixed time intervals. Clark proposed that the daily price changes of a futures contract belongs to a class of distributions characterized by trading volumes and subordinate to the normal distribution.

To test for normality, Clark estimated the Kurtosis statistic of cotton futures price changes. Kurtosis is the fourth moment divided by the variance of the distribution. If a distribution is normal, its Kurtosis will be 3. Otherwise its Kurtosis will depart from 3[9]. Clark grouped the daily price changes of the cotton futures according to the trading volume, and found the Kurtosis of the daily price changes of each group to be within the confidence interval of normal parent distribution which is about 3 standard deviations. Whereas the Kurtosis of the daily price changes of the whole sample is 100 standard deviations. This result indicates that each subgroup of the data is normally distributed, but the whole series is not normally distributed. The trading volumes used for grouping the data may reflect the number of transactions during the period and the rate at which new information arrives in the markets.

## 3.5 Martingale Model

The random walk model, which requires independence between successive price changes, is too restrictive. The weaker model, the martingale model, with the relaxation of the independence restriction still keeps the flavor of the random walk model. The martingale model was proposed by Samuelson (1965) for futures market price movements. But it also can be applied to other financial prices such as stock prices. The discussion below follows Samuelson's work and uses futures prices as the example.

For a spot price in the future, given the information of current and past spot prices, $\Phi_t = [Y_t, Y_{t-1}, \ldots]$, we can not know with certainty the future spot price $Y_{t+T}$. Suppose there is at best a probability distribution for any future spot price that depends

[9] See Taylor (1983), and Judge et. al. (1988), p.891.

solely on the number of periods in which we try to forecast spot prices. The probability of spot price being $Y_{t+\tau}$ is given by $P(Y,\Phi_t,\tau)$ with the property that:

$$P(Y,\Phi_t,\tau) = \int P(Y,Z,\Phi_t,\tau\text{-}1)\, dP(Z,\Phi_t,1) \qquad (3.5.1)$$

which means the probability of spot price being $Y_{t+\tau}$ at time $t+\tau$, given the information $\Phi_t$ at time t, is the sum of the joint probability of: (1) all possible spot price being $Z_{t+1}$ at time t+1 given $\Phi_t$ at time t; and (2) the spot price being $Y_{t+\tau}$ at time $t+\tau$ given the information $\{Z_{t+1},\Phi_t\}$ at time t+1. The equation (3.5.1) is very similar to that of Bachelier's random walk model with continuous time in equation (3.4.1).

Denote the futures price of the commodity quoted at time t for time $t+\tau$ as $y(\tau,t)$, and the futures price quoted at t+1 for time $t+\tau$ as $y(\tau\text{-}1,t+1)$, and so on. If we assume the futures price $y(\tau,t)$ is the expected value of the spot price $Y_{t+\tau}$, then:

$$y(\tau,t) = E[Y_{t+\tau}|\Phi_t] = \int Y\, dP(Y,\Phi_t,\tau) \;\text{ with }\; y(0,t+\tau)=Y_{t+\tau} \qquad (3.5.2)$$

If the spot price sequence $\{Y_t\}$ follows the laws of equation (3.5.1), and the futures price sequence $\{y(\tau,t), y(\tau\text{-}1,t+1), \ldots, y(0,t+\tau)\}$ follows the axioms of expected price as described by equation (3.5.2), then the futures price sequence follows a martingale model in the sense of having zero price changes,

$$E[\Delta^n y(\tau,t)] = 0\,, \qquad (3.5.3)$$

$$\Delta^n y(\tau,t) = y(\tau\text{-}n,t+n) - y(\tau,t)\ , \quad n = 1, 2 ,\ldots, \tau.$$

The equation (3.5.3) implies the expected price change of any futures contract is zero. Thus the expected returns on the futures contract is zero, and the martingale model of futures prices defined equation (3.5.3) satisfies the weak-form efficient market hypothesis.

The martingale model of futures prices suggests that the current price $y(\tau,t)$ contains all that can be known about future price movement in the sense that the expected price change is zero. However, the martingale model does not imply that the sequence of futures price conform to random walk and neither does it imply that $\Delta y(\tau,t)$ is statistically independent of $\Delta y(\tau+1,t\text{-}1)$. Samuelson remarked that it should be a source of comfort to economists that wheat prices do not perform a Brownian random walk which wanders indefinitely far.

To generalize Samuelson's discussion of the martingale model, we can look $y(\tau,t)$ as the price of a financial asset at time t. Replace $y(\tau,t)$ by $y_t$, then we can have:

$$E(\Delta y_{t+1} | \Phi_t) = 0\ ,$$

where $\Phi_t$ is the information of current and past prices. If we note that $\Delta y_{t+1} = y_{t+1} - y_t$, then we have:

$$E(y_{t+1} | \Phi_t) = y_t\ . \qquad (3.5.4)$$

And this is the expression used by LeRoy (1989) for the martingale model. This expression says that, if $y_t$ is a martingale, then the best (the unbiased) forecast of $y_{t+1}$ based on information set $\Phi_t$ is $y_t$.

If we want to discount the asset prices in the future to the present value, and consider the dividend of the asset given in period t+1, then we can rewrite equation (3.5.4) as:

$$(1+\rho)^{-1} E(y_{t+1}+d_{t+1}|\Phi_t) = y_t , \qquad (3.5.5)$$

where $\rho$ is the discount rate and $d_{t+1}$ is the dividend of the asset in period t+1. So the price of the asset itself is not a martingale, but the discounted price of the asset with dividend reinvested is a martingale. Some assets do not have dividend, so we can drop $d_{t+1}$ from equation (3.5.5).

One important difference between the martingale model and the random walk model is that the martingale model rules out any dependence of the conditional expectation of price changes, whereas the random walk model also rules out higher conditional moments of price changes. Thus if the variances of the price changes are correlated, the price changes can satisfy the martingale model, but they will not satisfy the random walk model. This types of serial dependence in variance will be discussed in Chapter 4.

Another important property of the martingale model is that it assumes agents are risk neutral. Because the asset which has large variance in its price changes also has large risk, if agents do not care what the higher moments of the price distributions are, as risk neutrality implies, then they will do nothing in the presence of serial dependence in the higher conditional moments of the price changes, or they will not require risk premium for risky assets. Even if there is serial dependence in the conditional variances it is irrelevant to the martingale model. Once researchers became aware of this property of the martingale model, they immediately realized that the efficient market models are leaning towards the martingale model rather than the random walk model. Researchers also realize that the tests of market efficiency are in fact the tests of martingale model, or the test of the weaker model that rates of return are uncorrelated.

### 3.6 Mean-Reverting Process

In testing and explaining the variability in stock prices, Shiller (1984) suggested that there may be a "fad" effect stock markets. A fad can reflect changes in attitude or fashion regarding investment in reaction to some widely known events. Summers (1986), in studying the market efficiency, noted that the persistent errors in market evaluation may be explained by fads in the market. The concept of fad in the market was further evolved into the mean-reverting process for the stock returns.

Then Poterba and Summers (1988) suggested in a more explicit way that the stock return series is the sum of random walks and a stationary mean-reverting process. They treated the logarithm of the stock price as the sum of permanent and transitory components. The permanent component follows a random walk and the transitory component follows a stationary process. The transitory component may reflect fads - the financially induced deviations of prices from fundamental values - or it may be a consequence of changes in required returns. Transitory components in stock prices imply variation in ex ante returns. If the market value and the fundamental value of the stock (derived from a company's capital assets, predicted earning, etc.) diverge beyond some range, the difference will be eliminated by financial forces, and the stock prices will revert to their mean. Returns must be negatively serially correlated at some frequencies if erroneous market moves are eventually corrected.

LeRoy (1989) further elaborated that when the stock prices comprise a random walk component and a slow varying fad variable, then the returns over the short interval will be uncorrelated but over the intermediate interval will be negatively correlated. This is because over the short intervals the random walk component dominates, but over the intermediate interval the fad component dominates. Over the intermediate interval, if the return in the previous period is high, than the fad is positive, mean-reverting implies that the fad will probably diminish in the current period. So over intermediate intervals the returns will be negatively correlated. But over very long period, the fad will diminish, and the negative correlation of the returns will also diminish. So as the interval of the correlation increases, the correlation of the returns decreases from zero to some negative value and then goes back to zero, drawing an U-shaped curve.

Poterba and Summers (1988) analyzed data on equal-weighted and value-weighted NYSE returns over the 1926 - 1985 periods, and data from other countries and time periods. Their results indicated that stock returns show positive serial correlation over short periods and negative serial correlation over longer intervals. Poterba and Summers found significant transitory price components. Transitory factors account for three-fourth of variance in some cases. Using a variance ratio test, they also found that over the long horizons, the variance ratio declines as horizon gets longer, which indicates the existence of mean-reverting process component in the market.

Fama and French (1988) found that stock prices show negative autocorrelation over long horizons beyond a year, where their study was also based on a variance-ratio test. As the horizon of the correlation increases, the graph of autocorrelations is U-shaped with the minimum occurring within at 3-5 year range. These are consistent with the mean-reverting process. In their study, Fama and French included stock returns of different industries and different group of firm sizes from the 1926 - 1985 period.

Lo and Mackinlay (1988) investigated the weekly returns of stocks using variance-ratio statistic. They showed that the weekly returns of stocks do not follow a random walk. They also showed that the results of variance-ratio statistic are inconsistent with the mean-reverting process. They concluded that the weekly returns do not fit the mean reverting model. Kim, Nelson and Startz (1988) calculated autocorrelation for the data used by Fama and French (1988), they found the evidence of mean-reverting only in the data sets that include the 1930s period, for the post-World War II period they found no evidence of negative autocorrelation in stock returns.

One point worth mentioning is that, Fama (1992) pointed out that for the work of Fama and French (1988), even with 60 years of data, the estimation of the autocorrelation of returns over long horizon (3 to 5 years) has small sample size and low power, and when the 1926-1940 period is deleted from the sample, the negative autocorrelation in 3 to 5 year return disappears. And for the work of Poterba and Summers (1988), even with 115 years of data, the variance-ratio test for returns over 2 to 8 years has small sample size and provides weak statistic evidence against zero autocorrelation and the random walk model.

The mean-reverting model suggested that the stock returns over short period follows a random walk and over long period follows a mean-reverting process. Lo and MacKinlay (1988) rejected the random walk component of the mean-reverting model, and Kim, Nelson and Startz (1988) rejected the mean-reverting component of the mean-reverting model. In a commentary to Shiller's (1984) paper, Benjamin M. Friedman also pointed out the presence of fads (or the mean-reverting, negative correlation over intermediate interval) may imply the predictability of the asset prices. But investors will not be able to predict the prices unless they know that the current period's fad is on the way in or out, and what the next period's fad will be. Because we can not predict the future fad, we will not be able to predict future prices either. Therefore the mean-reverting model also satisfies the weak-form efficient market hypothesis.

## 3.7 The Market Anomalies

The market anomalies, such as the "January effect" and the "days of the week effect", are departures from the models discussed in last few sections. In the following, I review the market anomalies found in financial prices and their relation to the efficient market hypothesis. The effect of market anomalies on modeling of financial prices will also be discussed in this section.

Rozeff and Kinney (1976) found that the average stock returns in January is 3.5 percent, and in other months is 0.5 percent. This January effect is inconsistent with the martingale model. Other subsequent studies also confirmed the January effect.[10] Banz (1983) found the "small firm effect" that the stocks of small firms have higher returns than is consistent with their riskiness. Keim (1983) showed that the small firm effect and the January effect may be the same thing: The January effect appears only in samples that include small firms and give equal weight to small and large firms, as opposed to samples that weight firms by value. In analyzing risk and returns trade-off, Tinic and West (1984) found that the risk-return trade-off occurs entirely in the month of January.

The days-of-the-week effect was first found in returns from close of trading on Fridays to the opening of trading on Mondays, thus the "weekend effect".[11] On average, the returns over the weekend are negative. Gibbsons and Hess (1981) also found the weekend effect exists in bond market. Later, the Wednesday effect was also noticed by researchers. For example, French and Roll (1986) found the variance from Tuesday to Thursday was lower than over other two-day period. They also found that on an hourly basis, the variance of price changes is 72 times higher during the trading hours than during the weekend non-trading hours, and is 13 times higher than during the overnight

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[10] For example, the works by Reiganum (1981, 1982), and Roll (1983).

[11] See, for example Cross (1973), French (1980), Keim and Stambaugh (1984), Harris (1986).

non-trading hours. These findings on the variance of price changes, however, may be caused by the difference in the flow of information during the trading and non trading hours.[12]

Chang and Kim (1988) studied the spot commodity price index and commodity futures price index, and found that the returns on Monday is lower than on other days, and the variance on Monday is larger than on other days. But they found that after 1982 the negative returns on commodity futures price index disappeared. Before 1981 the returns on Friday is the highest and after 1981 the return on Tuesday is the highest. They also found that the variances of the returns are not constant throughout the week.

Johnson, Kracaw and McConell (1991) reported the negative seasonal on Monday for GNMA and T-bond futures, and positive seasonal on Tuesday for GNMA, T-bond and T-note futures. But they also notice that the negative seasonal on Monday only in the data before 1982, the positive seasonal on Tuesday only in the data after 1984, and the weekly seasonal occur only during the months prior to a delivery month.

Martell and Trevino (1990) studied the intraday commodity futures and found that the behavior of intraday prices for a given contract is not homogenous over time. The intraday serial correlations of the futures prices gradually changes during the life of the contract from small positive values to negative values. The serial correlation of the prices is positive in inactive trading days, but is negative in the more active trading days.

Yang and Brorsen (1993) studied 15 futures price series from 1979 to 1988. They found the most prominent market anomalies is days-of-the-week effect in variance. The

[12] See the discussion in section (2.3), Workings (1953), and Clark (1974).

variance is larger on Mondays and after holidays. Several agriculture commodities showed seasonality in variances. A few commodities showed significant day-of-the-week in mean, or maturity effect in variance.

These market anomalies show that there are some patterns in financial prices. The market anomalies indicate the departure from the random walk model, the martingale model, and the mean-reverting model. Malkiel (1989) argued that the market anomalies found in financial prices are generally remarkably small and do not provide unexploited profitable opportunities. Therefore the market anomalies do not violate the weak-form efficient market hypothesis.

Thus when analyzing financial prices, if we detect serial dependence in the data, we first need to determine whether the serial dependence is caused by market anomalies. If market anomalies can not explain all the serial dependence, then we need to pursue further for the additional source of serial dependence. If market anomalies exist in the data and we fail to account them, we may have misspecification problem when fitting econometric model to financial prices.

## 3.8 Summary

The price changes in financial market are the concernce of economists, entrepreneurs, and ordinary consumers. In Chapter 6 of this dissertation we will study price movements in futures markets. Futures markets provide price information and hedging opportunities. But futures markets also share the same properties of other financial markets. They are speculative in nature and they are interrelated to other financial markets.

A fundamental hypothesis on the price movements in financial markets is the efficient market hypothesis. One version of the efficient market hypothesis is the weak-form efficient market hypothesis, which states that the asset prices reflect information contained in past prices, so it is impossible to formulate a profitable trading scheme based on past prices. Some theoretical and empirical studies have challenged the weak-form efficient market hypothesis. The random walk model, the martingale model, and the mean-reverting model are three models which satisfy the weak-form efficient market hypothesis.

The random walk model is the oldest model used for studying price movement in financial markets, it assumes the asset price changes (or log-price changes) are independently and identically distributed. Earlier empirical studies supported the random walk model. But many recent empirical studies rejected the random walk model. The martingale model is a less restrictive model than the random walk model. It just require that independence between past prices and the conditional expectation of price changes. In the martingale model the dependence between past prices and higher moment of price changes is allowed and the higher moments are irrelevant in the martingale model.

Another model in financial economics is the mean-reverting model, which assumes the financial prices have a random walk component and a mean-reverting process. There were some empirical support the mean-reverting model, but there were also some empirical evident against the mean-reverting model. Even if the financial prices follow the mean-reverting process, we will not be able to predict the future prices because we can not know the "mean" where the prices are reverting to. Thus the mean-reverting model satisfies the efficient market hypothesis.

The seasonal and daily market anomalies are discovered recently and they are generally attributed to calendar events. The calendar events thought as causing the anomalies are: tax reports, the weekend none trading days, etc.. The market anomalies indicate patterns in financial price series, but they are not large enough to yield profitable opportunities, and they do not contradict the weak-form efficient market hypothesis. When we conduct empirical study of price movements in financial markets we have to be aware of these market anomalies.

The models presented in this chapter have enhanced our understanding of price movements in financial markets. But those models also have their limitations and some empirical studies have rejected these models. And the traditional techniques for hypothesis testing of these models and the weak-form efficient market hypothesis are linear. So if the underlying process of price behavior in financial markets is nonlinear, these linear techniques will not be able to reveal the nonlinear process in financial prices. The recent research on nonlinear models and techniques is a major step forward for understanding price movements in financial markets. These topics will be discussed in the next chapter.

# CHAPTER 4

# TESTING AND MODELING NONLINEARITY IN TIME SERIES

## 4.1 Introduction

The financial economic models discussed in Chapter 3 have advanced our understanding of price dynamics in financial markets. The statistic methods used for studying these models were mostly linear. Recently, however, there have been increasing empirical evidence that raises questions about the appropriateness of these models and the techniques for studying them. Moreover, some new financial economic models were developed with nonlinear structure in them. Thus the use nonlinear econometric time-series models and techniques in studying price dynamics in financial markets appeared to be a potentially fruitful agenda.

Research in physical sciences has discovered new behavioral patterns in nonlinear dynamic systems. Among the interesting behavioral patterns, deterministic chaos has drawn special attention[1]. A nonlinear system can produce a deterministic time sequence (or path) that has the characteristics of stochastic process. A nonlinear system can also

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[1] Examples of this application includes turbulence and thermal convection in fluids, chemical reacting systems, climate behavior, biological population behavior, and other wide range of research in physics (laser, plasmas, solid state physics), See May (1975).

exhibit the appearance of unpredictability and structural change when in fact there is no shift in the model's structure at all. These findings have inspired researchers to use nonlinear time-series models in the study of price dynamics in financial markets.

The research in physical sciences also provides techniques for detecting nonlinear deterministic chaos.[2] But these methods are not statistical tests and they only deal with deterministic system. In financial economics we may encounter nonlinear stochastic time-series processes. Recently several empirical studies have indicated the existence of nonlinear stochastic process rather than nonlinear deterministic process in financial markets. Therefore we need new methods that can deal with nonlinear time-series processes as well as deterministic chaos and provide us with test statistics for study of financial prices. The new test statistics which will be studied in this dissertation are the BDS statistic, the TAR-F statistic, and the $Q^2$ statistic.

The BDS statistic is a new test statistic which can be used to detect nonlinear determinist process and nonlinear time series process. It is derived from the correlation integral. The null hypothesis of the BDS statistic is that the time series being independently and identically distributed (IID). If the time series is not IID, the BDS statistic will be able to detect the non-IID. The BDS statistic can also identify nonlinear time series when it is applied to linear fitted data of the time series. The BDS statistic can be used to test the adequacy of the forecasting model when applied to the forecasting errors of the model.

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[2] Such as the calculation of largest Lyapunov exponent and the calculation of dimension of the phase trajectory.

The other test statistics that can also be used for detecting nonlinearity are the TAR-F statistic which is designed for detecting threshold autoregressive nonlinearity, the McLeod-Li $Q^2$-statistic which is designed for detecting autoregressive conditional heteroskedasticity (ARCH) type nonlinearity, and the Bispectral test which has the ability to detect many types of nonlinearity. The first two test statistics are more interesting because they can be used for model identification in the model building process.

Two nonlinear econometric time-series models, the threshold autoregressive model and the ARCH-type model, are more attractive to financial economics. The threshold autoregressive model has nonlinearity in terms of conditional mean, while the ARCH-type model has nonlinearity in terms of changing variance of the time series. These two nonlinear models have been applied to economics and finance and have brought some improvement in the modeling of data.

In this chapter we review concepts of nonlinear deterministic dynamic system and the transition to chaos. Then we discuss how are the nonlinear determinist model and nonlinear time series model related to economics and finance. The BDS statistic, the TAR-F statistic, the $Q^2$-statistic, and the bispectral analyzes are be reviewed and discussed. Two nonlinear econometric time-series models, the threshold autoregressive model and ARCH-type model, and their estimation methods that will be used in this dissertation are also discussed.

## 4.2 Nonlinear Dynamics and Deterministic Chaos[3]

A dynamic system can be described using state variables. For some dynamic systems we have specific information on all the state variables, but for many other dynamic systems we may not know all the state variables. For example, a macro economic system may be described by GNP, investment, money supply, interest rate, unemployment rate, etc.. For the problem of price dynamics in commodity futures, the variables that describe this problem probably will be the price of the futures, the trading volume, the interest rate, the supply and demand of the commodity, the returns on other financial assets, and other financial and economic variables.

For a dynamic system, if $s_t$ is the S dimensional vector of state variable which contains all the information describing the system at time t, the space that contains all the possible paths of the state vector is called the state space and has dimension of S. This state vector has the information relevant to the behavior of the system during time t+1. So it follows that there must exist a function (law of motion) such that $s_{t+1} = f(s_t)$ for all t. In general we can not observe the state variable nor the function f, we can only observe a variable $x_t$ of the system at time t. In our current problem of price dynamics in financial markets, the observed variable $x_t$ is the log-price change of the asset. Because the state vector $s_t$ fully describes the dynamic system, the observed variable $x_t$ must depend on the state vector $s_t$. Hence there must exist a function g such that $x_t = g(s_t)$ for

[3] For detailed discussion of the topic, see May (1975), Ott (1981), Barnett and Chen (1989), Baumol and Benhabib (1989), and Savit (1989, 1990).

every t.

The dimension of the state space can be very large, and there can be many choices of state vectors $s_t$ and dynamical function f that can explain the observed variable $x_t$ deterministically. When study the observed variable and the underlying dynamic system, however, we should seek the simplest explanation for the observed data series and the lowest dimensional state space that can be used to produce a self-generating deterministic explanation of the past, present, and future behaviors of data series $x_t$.

The observed variable $x_t$ combined with the state vector $s_t$ will produce a system vector of dimension S+1. In general each variable of the combined system vector evolves continuously in accordance with a differential equation of first order in time:

$$\dot{x} = f^0(x,s^1,\dots,s^S) \ ,$$
$$\dot{s}^i = f^i(x,s^1,\dots,s^S) \ , \quad i=1, \dots, S \ , \qquad (4.2.1)$$

where the dot over the variable denotes the time derivative. Theoretically, by repeated differentiation in time and by substitution, the S+1 differential equations can be reduced to a single differential equation of order S+1 in the observed variable $x_t$. Hence we see that the observations on only one variable, $x_t$, are sufficient to permit us to go beyond its own one-dimensional space to infer information about the dynamic system f itself, which is defined over the unknown state space.

For a nonlinear dynamic system, the variable $x_t$ can evolve in a remarkable number of ways. Guckenheimer and Holmes (1983) observed that "simple differential equations of dimension three or greater can possess solutions of stunning complexity." The variable $x_t$ can converge to a constant or a cycle, or evolve to deterministic chaos having the properties of almost any kind of stochastic process[4]. A deterministic system that can produce chaotic time sequence (or path) is called deterministic chaotic dynamic system. A time sequence is chaotic, or "turbulent," if the sequence has the following properties: (1) sensitive dependence on the initial condition (called the seed); (2) a form of stationarity, and; (3) nonperiodicity.

The points in the state space that draw the sequence of state vector $\{s_t\}$ towards them if the initial point of the sequence is sufficiently close to them is called attracting points. The region where if the initial point starts and the sequence $\{s_t\}$ will be drawn by the attracting points is called the basin of the attractor. The behavior of the sequence of state vector $\{s_t\}$ is determined by the geometry of the attractor. The attractor which contains one attracting point is called an attractor of period one. If there are two attracting points in the system, then the point near them will be simultaneously attracted by two separate points. The attractor which has two attracting points is called an attractor of period two. The attractor of period one will become an attractor of period two when the parameter characterizing the model changes and passes a critical value, and this critical value is called bifurcation point.

To illustrate, consider an economic system of Cobweb model in which the quantity demanded in the current period is a function of the current price, the quantity supplied in the current period is a function of last period's price, and the supply is equal

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[4] See Ruelle and Takens (1971), Li and Yorke (1975).

to the demand, that is:

$$q^D_t = f^D(\alpha, p_t) ,$$

$$q^S_t = f^S(\beta, p_{t-1}) ,$$

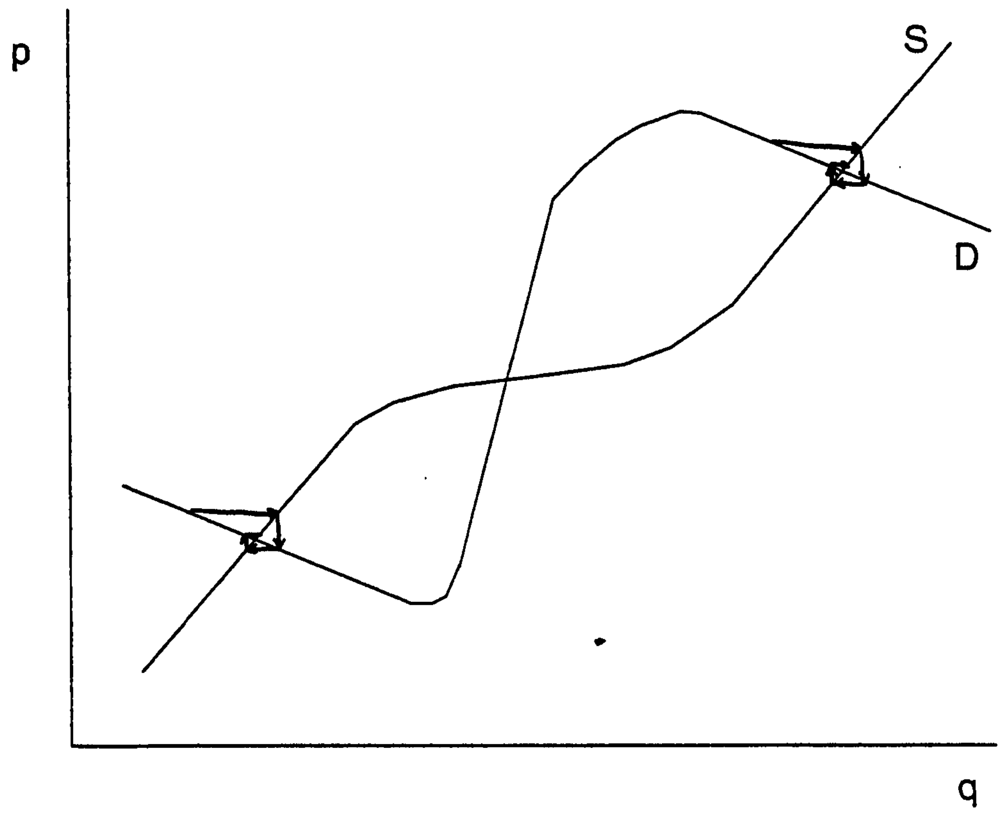
$$q^D_t = q^S_t ,$$

where $q^D_t$ and $q^S_t$ are respectively the quantity demanded and the quantity supplied in the period t, $p_t$ is the price in period t, $\alpha$ and $\beta$ are the parameters in demand function and the supply function. The shapes of the supply curve and the demand curve, defined by the supply function and the demand function, can be varied by the parameter $\beta$ and $\alpha$ respectively. If the supply curve and demand curve cross each other, and at the crossing point the slope of supply curve is greater than the slope of demand curve, then the crossing point is the equilibrium point and the attracting point (see Figure 4.1). If the initial price is close to but not at the equilibrium point, then it will move towards to and eventually reach the equilibrium point. Suppose as one or both parameters of the demand function and supply function change, the supply and demand curve cross more that once, there can be more than one attracting point in the model.

For a dynamic system, the most noted way of transition to chaos is through the bifurcation of period doubling, which is explained by the following. As a parameter of the model changes, the attractor with one attracting point will evolve to an attractor with two attracting points. When the parameter of the model changes further, the two attracting points of the attractor will separate farther and farther apart. Eventually, each

Figure 4.1

Cobweb Model



attracting point will bifurcate into two attracting points. The number of attracting points doubles every time when bifurcation occurs, since at bifurcation each attracting point spans its own new two attracting points and every attracting point in the attractor bifurcates simultaneously. Hence, bifurcation produces period doubling. As parameter changes and passes from one bifurcation point to the next bifurcation point, the distance between the two successive bifurcation points gets smaller and smaller, with that distance being exactly predictable from a convergence rate constant called Feigenbaum's number[5]. The attractor also can start from a loop (limit cycle), and then bifurcate to an n-periodic, n-torus attractor.

At the limit, after infinite occurrence of bifurcations, the attractor set can have infinite number of points. If the attractor set forms a line, or a square, or a cubic, then it will have dimension of 1, or 2, or 3 respectively. If the attractor set has a fractional (non integer) dimension then it is called fractal[6], and we say that the attractor is a strange attractor. The existence of a strange attractor is neither necessary nor sufficient for the system being chaotic. Nevertheless, in virtually all studies of chaos, a strange attractor exists. Hence, we consider the existence of chaotic dynamics as the existence of a strange attractor drawing the time path toward it.

The attractor set is a subset of the state space. Typically we have no measured value of the state vector $s_t$, we only have a measured value of the observed variable $x_t$.

---

[5] Suppose $\{\alpha_i\}$ is the sequence of the parameter where the bifurcation occurs, then the Feigenbaum's number is defined as $(\alpha_i-\alpha_{i-1})/(\alpha_{i-1}-\alpha_{i-2})$ when $i \to \infty$.

[6] See Mandelbrot (1977).

Hence, it is important for us to have a means of relating the behaviors of state vector $s_t$ to the behaviors of observed variable $x_t$. Takens (1980) has done so through an embedding theorem, which can be understood as follows. Select a state vector from the basin of the attractor; produce an n-history (orbit) of $x_t$ created by n-1 iterations on (g,f) (i.e., $x_t = g(s_t)$, and $s_t = f(s_{t-1})$); then stack the resulting n values of $x_t$ into an n-history. If the first observation in the n-history is $x_t$, we can designate an n-history as following:

$$x(n)_t = (x_t, x_{t+1}, \ldots, x_{t+n-1}) \quad . \tag{4.2.2}$$

The space of those n-histories $x(n)_t$ for a fixed choice of n is called the phase space, and the selected value of n is called the embedding dimension. The set of phase space trajectories for all possible initial conditions for the state vector within the basin set is called the phase portrait of the system.

The choice of the embedding dimension n is very important for us to unfold and reveal the structure of a complex dynamic system. To study the properties of the attractor, the embedding dimension must be selected to exceed the dimension of the attractor.

Suppose the construction of $x(n)_t$ for a fixed embedding dimension is repeated for each possible value of $s_t$ within the basin set. Takens proved that there exists a deterministic dynamic system F in the phase space for any fixed embedding dimension n such that a one-to-one correspondence exists between the dynamic properties (in particular, all conjugate invariant) of $s_{t+1}=f(s_t)$ in the state space and the dynamical properties of $x(n)_{t+1}=F(x(n)_t)$ in the phase space. Those invariant properties include the Lyapunov exponent as well as dimension and entropy concepts.

Two of popular techniques in physical science to detect deterministic chaos are the computation of the attractor's dimension and calculation of the orbit's Lyapunov exponent. These two techniques also can be used in economics and finance because of the relatively easy calculations involved[7]. But these two techniques require large number of observation and are not statistical tests. And these techniques are not intended to deal with nonlinear time series process where misleading conclusion could be obtained.[8] The nonlinear statistical tests discussed in the next few sections are the techniques that can fill this gap.

## 4.3 Nonlinearity and Price Dynamics in Financial Markets

For the problem of price dynamics in financial markets, suppose the price dynamics $x_t$ are governed by a S-dimensional state vector $s_t$ and by the underlying dynamics f, such that:

$$s_t = f(s_{t-1}) \quad ,$$
$$x_t = g(s_t) \quad . \qquad (4.3.1)$$

---

[7] See, for example, Brock and Sayers (1985), Barnett and Chen (1986), Blanck (1990), Yang and Brorsen (1991).

[8] For example, Liu (1989) showed that the noise in the model can cause over estimation of the correlation dimension.

State vector $s_t$ consists of variables which we do not have observation. Following the discussion of last section, we can construct an n-dimensional vector from the observed variable:

$$x(n)_t = (x_t, x_{t+1}, \dots, x_{t+n-1}) \quad . \tag{4.3.2}$$

So there will be a dynamic system F such that:

$$x(n)_t = F(x(n)_{t-1}) \quad . \tag{4.3.3}$$

And the dynamic properties of F will be the one-to-one image of the dynamic properties of f. In particular, if f is nonlinear, then F will also be nonlinear.

For the n-th component of $x(n)_t$ in equation (4.3.3), we have:

$$x_{t+n-1} = F_n(x_{t-1}, x_t, \dots, x_{t-1+n-1}) \quad . \tag{4.3.4}$$

In equation (4.3.4), if we rewrite t+n-1 as t, $F_n$ as G, and rearrange orders of $x_i$'s in function f, then we have:

$$x_t = G(x_{t-1}, x_{t-2}, \dots, x_{t-n}) \quad . \tag{4.3.5}$$

So the current price change can be expressed as the function of past price changes.

Equation (4.3.5) also can be thought of as a model with a feed back mechanism in the price dynamics when the current price changes are influenced by past price changes. If we add an error term (or noise) to the deterministic model in equation (4.3.5), it becomes a stochastic model which has more features than the deterministic model. Thus the deterministic model is a special case of the stochastic model when the error terms are identically zero.

When price changes follow equation (4.3.5), the price changes will not be a random walk. Nevertheless, when the equation (4.3.5) is nonlinear, the resulting price changes can show deterministic chaos with random walk appearance. Moreover, because random walk model have been rejected by recent empirical tests, the nonlinear model becomes more appealing for study price dynamics in financial markets.

A popular nonlinear dynamic model is the logistic model[9], which is a nonlinear process with one period lag (see Figure 4.2). Suppose the price change in the current period is influenced by the price change in last period in a way such that

$$x_{t+1} = b\, x_t\, (1 - x_t) \ , \qquad (4.3.6)$$

where b is a constant parameter, and the price changes $x_t$ vary within the interval (0, 1).

In the logistic model, when the parameter b is less than 1, the price changes starting at any initial point will converge to zero. Thus zero is the only attracting point

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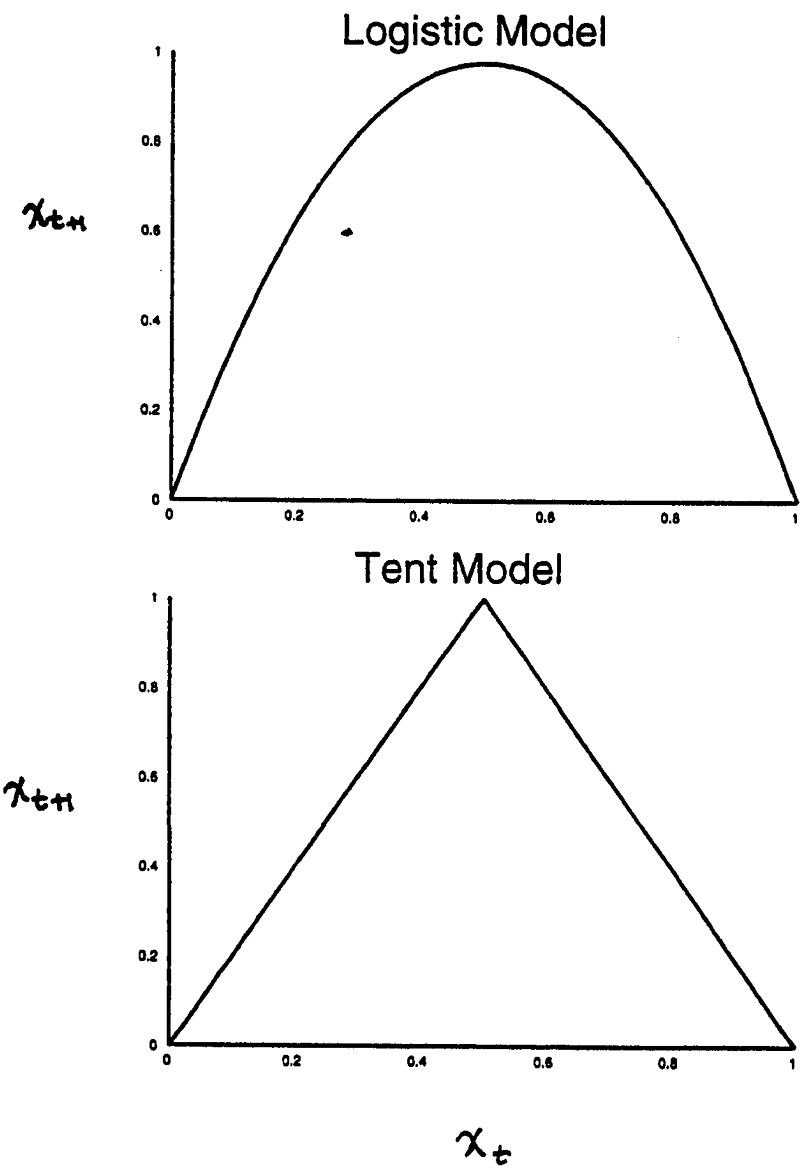
[9] See, for example, the discussion in physics by Ott (1981), Grassberger and Procaccia (1983), and in finance by Savit (1989).

Figure 4.2

Logistic Model and Tent Model



when b < 1. When 1 < b < 3, the time series of price changes will be drawn by 2 non-zero attracting points. As the value of parameter b increase further and further, the price changes will subsequently be attracted by 4 attracting points, 8 attracting points, and so on. After the value of parameter b exceeded 3.5699, the price changes will be attracted by infinite number of attracting points. Thus the price changes will move in a deterministic chaotic way when b > 3.5699. To illustrate the deterministic chaos generated by logistic model, Figure 4.3 shows the sequence generated by logistic model and the log-price changes of S&P 500 index futures.

Another simple nonlinear dynamic model is the tent map (see Figure 4.2). The tent map is defined by:

$$x_{t+1} = 2\,x_t\,, \quad \text{if } x_t \leq 0.5\;,$$
$$= 1\text{-}2\,x_t\,, \quad \text{if } x_t > 0.5\;. \tag{4.3.7}$$

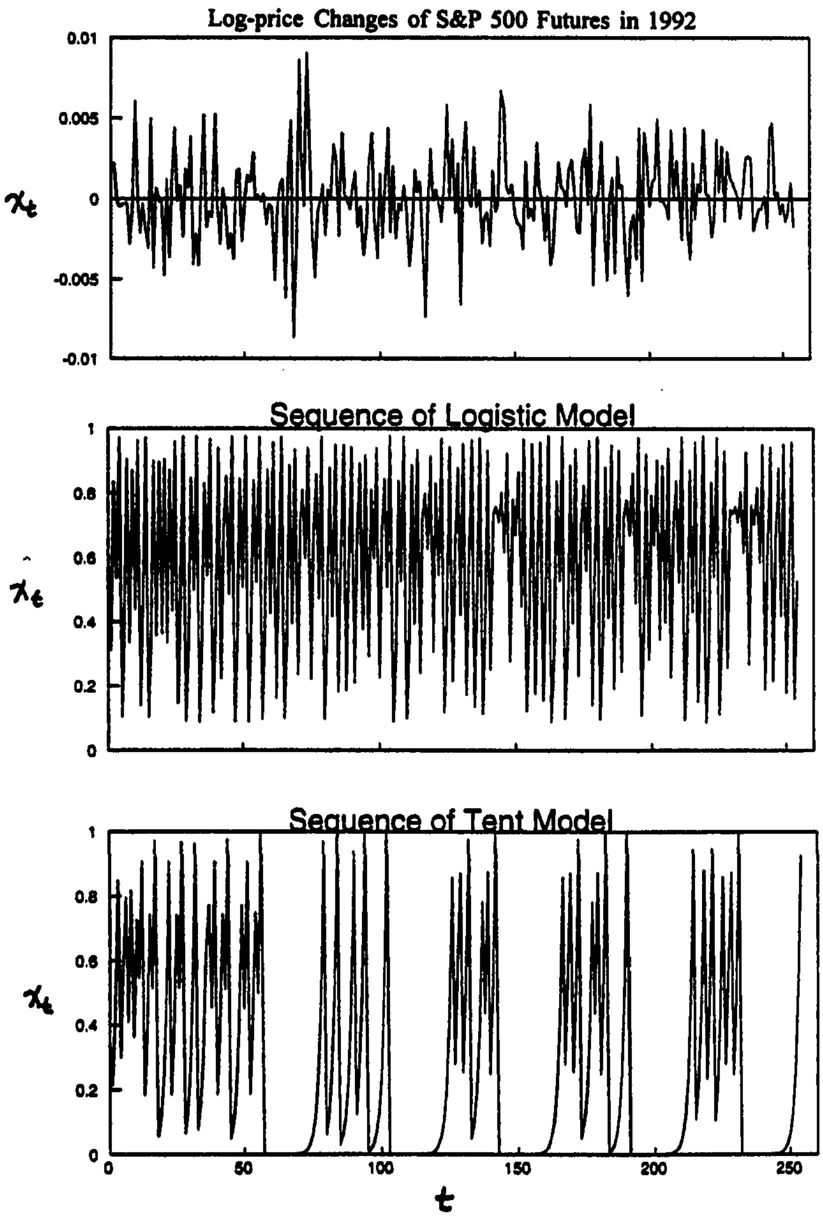
The variable $x_t$ is restricted in the region (0, 1). The map from $x_t$ to $x_{t+1}$ looks like a tent, so this nonlinear model is called tent map. The sequence $x_t$ generated from this model is chaotic, and the more detailed discussion of this model can be found in the literature such as Baumol and Benhabib (1989). Figure 4.3 shows a sequence generated by the tent map.

The logistic model and tent model discussed here are just two simple examples of nonlinear discrete dynamic models which show features of deterministic chaos and the potential to model price dynamics in financial markets. This does not imply that they are

Figure 4.3

Log-Price Changes of S&P 500 Futures and

Sequences Generated by Logistic Model and Tent Model



the models which actually describes price dynamics in financial markets.

Other nonlinear models were also developed in economics and finance. For example, Benhabib, Day, and Grandmont (Grandmont, 1985) had considered an overlapping generation model. In this model the two generations maximize their utilities subject to budget constrains. The resulting dynamic function of real balance of the young generation is nonlinear and can lead to deterministic chaotic behavior. Savit (1989) also showed that the option prices can have nonlinear deterministic chaotic component, such as the tent map in the formulation of option prices.

Several empirical studies were attempted to detect low dimensional nonlinear deterministic chaos in finance. Blank (1990) studied the price changes in futures markets, and concluded there are evidence of nonlinear deterministic process. Peters (1991) studied the returns of S&P 500 index and indicated the existence of deterministic chaotic attractor. Hsieh (1991) studied the returns of several stock indices, but rejected the existence of nonlinear deterministic model. Yang and Brorsen (1992 and 1993) studied the commodity prices and futures prices, and rejected the hypothesis of nonlinear deterministic process. The studies of Hsieh (1991) and Yang and Brorsen (1992 and 1993) suggested the nonlinear stochastic time-series model for modeling price dynamics in financial markets.

Some theoretical works in economics and finance also yielded nonlinear stochastic models. For example, Aiyagari, Eckstein, and Eichenbaum (1985) studied the prices of storable goods, and found that the prices of storable good will switch between two linear stochastic processes when the inventory changes from positive to zero or from zero to positive. This model of storable good prices is a threshold autoregressive model of Tong and Lim (1980). Hsieh (1988) also formulated a nonlinear stochastic rational expectations model of exchange rate under central bank interventions. The model of exchange rate is also a nonlinear switching model similar to threshold autoregressive model. Lai and Pauly (1988) also formulated a model for foreign exchange rates, where the equilibrium exchange rates is an ARMA process with conditional heteroskedastic errors. The nonlinear econometric models are also useful for explaining behavior of economic and finance data. Two of the most useful nonlinear econometric models are discussed in detail in Section 4.5 and Section 4.6.

## 4.4 The BDS Statistic

The BDS statistic proposed by Brock, Dechert, and Scheinkman (1987) is based on correlation integral. It is developed to avoid the short comings of the Grassberger-Procaccia method of detecting deterministic chaos using correlation dimension. Basically the BDS statistic is for testing whether a time series satisfies the independently and identically distributed assumption. But it can also be used to detect nonlinearity in the time series when applied to the residuals of linear filtered model. The BDS statistic is particularly useful because it can detect nonlinear serial dependence in the time series where the linear technique of autocovariance generally fails.[10]

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[10] Sakai and Tokumaru (1980) showed that certain deterministic chaotic time sequences such as tent map can have zero autocorrelation. Bunow and Weiss (1979) also demonstrated that the very simple deterministic chaotic model of tent map can generate

For time series $\{x_t\}$, at embedding dimension of M, its M-histories $x(M)_i$ is defined by:

$$x(M)_i = (x_i, x_{i+1}, \ldots, x_{i+M-1}) , \tag{4.4.1}$$

and its correlation integral $C_{M,T}(L)$ at embedding dimension of M and correlation length L (in the unit of sample standard deviation) is:

$$C_{M,T}(L) = \frac{2}{(T-M+1)(T-M)} \sum_{1 \le i < j \le T} I_L(x(M)_i, x(M)_j) \tag{4.4.2}$$

where T is the sample size, and the indicator function $I_L(.)$ is defined by:

$$I_L(x(M)_i, x(M)_j) = 1 \quad \text{if sup norm } \|x(M)_i - x(M)_j\| < L ,^{11}$$

$$I_L(x(M)_i, x(M)_j) = 0 \quad \text{elsewhere} \quad . \tag{4.4.3}$$

The BDS statistic of the time series at embedding dimension M, correlation length L, and sample size T is:

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time paths, autocorrelation functions and spectral power density functions appearing to be indistinguishable from those generated by pseudo-random numbers.

[11] For an n-dimensional vector $y = (y^1, y^2, \ldots, y^n)$, sup norm $\|y\| = \max |y^i|$, $i = 1, 2, \ldots, n$, which is largest absolute value of y's coordinate.

$$w_{m,T}(L) = T^{1/2}(C_{M,T}(L)-C_{1,T}(L)^M)/\sigma_{M,T}(L) \,, \qquad (4.4.4)$$

where:

$$\sigma^2_{M,T}(L) = 4[K^M+2\sum_{j=1}^{M-1} K^{M-j}C_{1,T}(L)^{2j}+(M-1)^2C_{1,T}(L)^{2M}-M^2KC_{1,T}(L)^{2M-2}] \qquad (4.4.5)$$

$$K = \frac{6}{(T-M+1)(T-M)(T-M-1)}\sum_{1\le i<j<k\le T} h_L(x(M)_i,x(M)_j,x(M)_k) \qquad (4.4.6)$$

$$\begin{aligned} h_L(.) = {} & (I_L(x(M)_i,x(M)_j)I_L(x(M)_j,x(M)_k)+I_L(x(M)_i,x(M)_k)I_L(x(M)_k,x(M)_j) \\ & +I_L(x(M)_j,x(M)_i)I_L(x(M)_i,x(M)_k))/3 \end{aligned} \qquad (4.4.7)$$

Hsieh (1989) provided a intuitive explanation of the BDS statistic as follows. The correlation integral $C_{M,T}(L)$ is the estimation of the probability that any two M-histories $x(M)_i$ and $x(M)_j$ are within the distance of L from each other, and we have:

$$C_{M,T}(L) \text{ converges to } \text{prob}\{\|x(M)_i-x(M)_j\| < L\} \text{ as } T \to \infty \,.$$

If x's are independent, then the joint probability is the product of the probabilities of individual event, therefore:

$$\text{prob}\{\|x(M)_i-x(M)_j\| < L\} = \prod_{k=0,M-1} \text{prob}\{\|x_{i+k}-x_{j+k}\| < L\}$$

$$C_{M,T}(L) \text{ converges to } \prod_{k=0,M-1} \text{prob}\{\|x_{i+k}-x_{j+k}\| < L\} \text{ as } T \to \infty \,.$$

If x's are identically distributed, then $\text{prob}\{\|x_{i+k}-x_{j+k}\| < L\}$ will be same for all k. Thus we have:

$$\prod_{k=0,M-1} \text{prob}\{\|x_{i+k}-x_{j+k}\| < L\} = [\text{prob}\{\|x_i-x_j\| < L\}]^M .$$

Because $C_{1,T}(L)$ converges to prob$\{\|x_i-x_j\| < L\}$, so $C_{M,T}(L)$ converges to $C_{1,T}(L)^M$ as $T\rightarrow\infty$. Therefore $T^{1/2}(C_M(L)-C_1(L)^M)/\sigma_{M,T}$ has the standard normal distribution when $\{x_i\}$ is independently and identically distributed with $\sigma_{M,T}$ given by equation (4.4.5).

The null hypothesis of the BDS statistic is that the time series is IID. Brock, Dechert and Scheinkman (1987) showed that if a time series is IID, then the BDS statistic is distributed asymptotically standard normal. For the alternative of non-IID time series, the BDS statistic will not be standard normal distributed. When testing a time series, we can compute its BDS statistic and use critical values of the standard normal distribution to decide the acceptance or rejection of the IID null hypothesis.

The BDS statistic can detect non-IID time series, include linear and nonlinear time series. In order to use the BDS statistic for detecting nonlinear time series, we apply the BDS statistic to the linear filtered time series. If the original time series is linear, the filtered time series will be IID. If the original time series is nonlinear, then the filtered time series will not be IID. Therefore, when applied to the linear filtered time series, the BDS statistic can detect nonlinear time series[12].

The BDS statistic also can be used to test the adequacy of forecasting model. If we have a forecasting model, we can apply the BDS statistic to the forecasting errors of

[12] See Brock (1987) for the proof of the theorem.

the model. If there is no remaining forecastable structure, then the forecast errors should pass the BDS test. If the forecast errors are rejected as non-IID by the BDS test, then there is some forecastable structure in the forecast errors, such as serial dependence or changing variance. In this case we need to identify the forecastable structure and formulate a new model.

The asymptotic distribution of the BDS statistic is standard normal under the IID null hypothesis. Hsieh and LeBaron (1988) showed that the asymptotic distribution of the BDS statistic under the alternative hypothesis is a normal distribution with unknown mean and unknown variance. Furthermore the finite sample distribution of the BDS statistic under both the null and alternative hypothesis can not be derived analytically. Therefore in the empirical application of the BDS test we only have the asymptotic distribution of the BDS statistic under the null hypothesis for interpreting the test results.

In order to learn the finite sample distribution of the BDS statistic, Brock, Dechert and Scheinkman (1987) did a Monte Carlo study of the BDS statistic on a small scale with a single sample size, a single fixed parameter value, and few types of time series. Hsieh and LeBaron (1988), Brock, Hsieh and LeBaron (1991) expanded the scale of Monte Carlo experiments on the BDS statistic with more sample sizes and more types of time series. Of the most recent study, Brock, Hsieh and LeBaron (1991) presented estimations of finite sample distribution of the BDS statistic for several IID and non-IID time series. They found the estimated distribution of the BDS statistic of the IID time series studied approximate the asymptotical distribution when the embedding dimension M, correlation length L and sample size T are in reasonable ranges, i.e., $M<5$, 1.0<L<2.0, and T>500. For those non-IID time series studied, with the parameters of the time series (such as AR coefficient) fixed at a relatively large value, Brock, Hsieh, and LeBaron (1991) estimated the distribution and rejection frequency of the BDS statistic. The results showed that under the alternative hypothesis the distribution of the BDS statistic departs from standard normal and has power to reject null hypothesis. But still we do not know the finite sample distribution and power of the BDS statistic when the parameters of the non-IID time series take other values, especially when the values of parameters are small representing the weak serial dependence in the time series. These aspects of the BDS statistic will be studied in Chapter 5.

Scheinkman and LeBaron (1989) applied the BDS statistic to US GNP data and growth rate of industrial production, they found the existence of nonlinearity in the data. Hsieh (1989, and 1991) also used the BDS statistic to test five daily currency exchange rates from 1974 to 1983 and the returns of S&P 500 indices. The BDS statistic rejected IID null hypothesis for all the exchange rates and the returns of S&P 500 indices. Yang and Brorsen (1992, 1993) applied the BDS statistic to 9 commodity spot prices and 15 futures prices from 1979 to 1988. In all cases, the BDS statistic rejected the IID null hypothesis.

## 4.5 Threshold Autoregressive Model and TAR-F Test

Threshold autoregressive (TAR) model was first developed by Tong and Lim (1980). The TAR model is a piece-wise linear model, the linear process to be followed in the current period depends on which threshold region does a lagged value of the time series fall into.

Specifically, a time series $x_t$ is a self-exiting threshold autoregressive process if it follows the model:

$$x_t = \alpha^{(j)}_0 + \sum_{i=1,p} \alpha^{(j)}_i x_{t-i} + \epsilon^{(j)}_t\,, \quad r_{j-1} \leq x_{t-d} < r_j\,, \qquad (4.5.1)$$

where $j=1,\ldots,k+1$, k is the number of threshold, p is the AR order, d is the threshold lag, and $x_{t-d}$ is the threshold variable. The threshold values of the TAR model are $-\infty = r_0 < r_1 < \ldots < r_k < r_{k+1} = \infty$; for each j, $\{\epsilon^{(j)}_t\}$ is IID with zero mean and variance $\sigma_j^2$. The TAR model partitions the one-dimensional Euclidean space into k+1 regions and follows a linear AR process in each region. The overall model for $x_t$ is not linear when there are at least two regions with two different linear processes. Tong (1978, 1983), Tong and Lim (1980) proposed this nonlinear time series model as an alternative model for describing periodic time series. The TAR model has certain features such as limit cycle, amplitude dependent frequencies, and jump phenomena, that can not be captured by a linear time series model. For instance, Tong and Lim (1980) showed that the TAR model is capable of producing asymmetric, periodic behavior exhibited in the annual Wolf's sunspot data and the Canadian LYNX data.

The TAR model is generally characterized by the threshold lag d, the number of threshold k, the threshold values $r_j$, the AR order p, and the AR coefficients $a_i^{(j)}$ in each threshold region. A simple TAR model is the following:

$$x_t = \alpha x_{t-1} + \epsilon_t, \text{ if } x_{t-1} < 0,$$
$$= -\alpha x_{t-1} + \epsilon_t, \text{ if } x_{t-1} \geq 0. \quad (4.5.2)$$

In this model, the threshold lag is 1, the number of threshold is 1, the threshold value is 0.0, the AR order is 1, and the AR coefficients are $\alpha$ in first threshold region, $-\alpha$ in second threshold region. If the noise term $\epsilon_t$ in this model is zero, then the model can represent a deterministic nonlinear chaotic model, the tent map. This simple TAR model also can be used to approximate the nonlinear time series process such as:

$$x_t = - x_{t-1}^2/(1 + x_{t-1}^2) + \epsilon_t. \quad (4.5.3)$$

And theoretically other nonlinear time series models such as $x_t = f(x_{t-1}) + \epsilon_t$ can be approximated by a TAR model if we use sufficiently large number of threshold values at each turning point of the function f.

For detecting the TAR process, several tests have been developed, notably the test of Tong (1983), the portmanteau P-test of Petrucelli and Davies (1986), and the TAR-F test of Tsay (1989). The TAR-F test is used in this dissertation because the TAR-F test statistic can also be used in the model building procedure of the TAR model. Following is a brief discussion of the TAR-F test and the model building procedure of TAR model proposed by Tsay (1989).

To simplify the discussion, suppose the time series has a non-trivial threshold $r_1$ and the threshold lag d. Then we do an arranged regression, which is arranged by ascending order of the threshold variable $x_{t-d}$. Suppose we start a recursive regression with first m observations in the arranged data. When m is not so large such that all threshold variables of the m observations are in the first threshold region, the estimates of the coefficients are consistent if there are sufficiently large numbers of observations in the regression. In this case, the predictive residuals are white noise asymptotically and orthogonal to the regressors $\{x_{t-i} \mid i=1,\ldots, p\}$. When m is too large such that some threshold variables are in the second threshold region, then the corresponding predictive residuals will be biased because the model changed. And the predictive residuals will depend on the regressors $\{x_{t-i} \mid i=1,\ldots, p\}$. Consequently the orthogonality between the predictive residuals and the regressors will be destroyed once the recursive regression proceeds to the observations whose threshold variable $x_{t-d}$ exceeds $r_1$. Here the actual value of $r_1$ is not required, all that is needed is its existence.

To calculate the TAR-F statistic, we select an AR order p and a threshold lag d. The regressor of the regression is $\{(1,x_{t-1}, \ldots, x_{t-p}) \mid t=p+1, \ldots, T\}$. We arrange the observation $x_t$ and the aggressor $(1,x_{t-1}, \ldots, x_{t-p})$ by the ascending order of the threshold variable $x_{t-d}$. After the arrangement, the n-th observation is $x_{\pi n}$ and its regressor is $(1,x_{\pi n-1}, \ldots, x_{\pi n-p})$. Now the threshold variable is $x_{\pi n-d}$, and $x_{\pi(n-1)-d} \leq x_{\pi n-d}$. Then for the arranged regression, we use first b observations to do least square estimation. Then we use the remaining observation to do recursive regression. Suppose $A_m$ is the estimate of the coefficient based on first m observations, $P_m$ is the associated X'X inverse matrix, $q_m$ is the regressor. Then the recursive least squares estimates can computed efficiently by:

$$A_{m+1} = A_m + K_{m+1}[x_{\tau(m+1)} - q'_{m+1}A_m] ,$$

$$D_{m+1} = 1.0 + q'_{m+1}P_m q_{m+1} ,$$

$$K_{m+1} = P_m q_{m+1}/D_{m+1} ,$$

$$P_{m+1} = (I - P_m q_{m+1} q'_{m+1}/D_{m+1})P_m . \quad (4.5.4)$$

The standardized predictive residuals of the arranged recursive regression are given by:

$$e_{\tau(m+1)} = [x_{\tau(m+1)} - q'_{m+1}A_m]/D_{m+1}^{1/2} . \quad (4.5.5)$$

For selected p and d, the effective number of observation in the arranged regression is T-d-h+1, where T is the sample size, h=max{1,p+1-d}. Because we start the recursive regression with b observations, the recursive regression has T-d-b-h+1 standardized predictive residuals. Now perform the following least square regression:

$$e_{\pi i} = \omega_0 + \sum_{v=1,p} \beta_v x_{\pi i\text{-}v} + \epsilon_{\pi i} , \quad i=b+1, \ldots, T\text{-}d\text{-}h+1, \quad (4.5.6)$$

and compute the associated F statistic, the TAR-F statistic:

$$F(p,d) = \frac{(\sum e_t^2 - \sum \epsilon_t^2)/(p+1)}{\sum e_t^2/(T-d-b-p-h)} \quad (4.5.7)$$

Tsay (1989) proved that if $x_t$ is a linear stationary AR process of order p, then for large T, F(p,d) follows approximately an F distribution with p+1 and T-d-b-p-h degrees of freedom. Further more, $(p+1)F(p,d)$ is asymptotically a chi-squared random variable with $p+1$ degrees of freedom.

For a $F_{u,v}$ distribution, its mean and variance are:

$$\text{mean}(F_{u,v}) = v/(v-2)\ , \tag{4.5.8}$$

$$\text{var}(F_{u,v}) = 2v^2(u+v-2)/[u(v-2)^2(v-4)]\ . \tag{4.5.9}$$

So in the Monte Carlo study of the TAR-F statistic we can use these results to compare the simulated TAR-F statistic.

The null hypothesis of the TAR-F test is that the time series is a linear AR process. Under the null hypothesis, the TAR-F statistic will have an F distribution. Under the alternative, if the time series does not follow the same linear model when the threshold variable moves into different regions, then the TAR-F statistic will not have an F distribution. And we can use the critical values of the corresponding F distribution for rejecting or accepting of null linear AR hypothesis.

So for the study of finite sample behavior of the TAR-F test has been limited. Using Monte Carlo method, Tsay (1989) examined the TAR-F test on some data generating processes (DGPs) of TAR model with $d=1$, $k=1$, $p=1$, and found that the TAR-F test has power of detecting TAR type nonlinearity. Tsay also found that the TAR-F test does not give high rejection frequency for DGPs of linear AR. In another study, Tsay (1991) investigated the TAR-F test on several types of nonlinear DGPs and concluded that the TAR-F test has good power rejecting them. However, in these studies, the number of sample size and number of parameter values of DGP are few. In Chapter 5, the TAR-F test will be studied with more sample sizes and more parameter values of DGP.

Tsay (1989) also provided a procedure for building the TAR model. In order to specify the threshold lag d, we should run the test for different threshold lags with a AR order p not too small, select the threshold lag which has the largest TAR-F statistic. For identifying threshold values of the r's, we should run the arranged recursive autoregression again with the specified threshold lag, get the scatter-plot of t-ratios of the AR coefficients at each recursive stage. If the arranged autoregression passes a threshold value by even a short distance, then the pattern of gradual convergence of the t-ratio will be destroyed and the t-ratio will have a big change. The places where the t-ratio has big changes are the locations of the threshold values. Once the threshold values are determined, we should select the AR order in each threshold regime by using Akaiki information criteria (AIC). We can also do a fine tuning of threshold values by varying threshold values and minimizing overall AIC values.

Although TAR model is useful for analyzing finance data, its application so far has been limited. Tsay (1989) studied the sunspot date from 1700 to 1979, the Canadian LYNX data, and the hourly attic temperature. The TAR-F test indicated the existence of threshold in all three cases. When the TAR model is applied to these three group of data, the fitting of the data were improved. Geweke and Terui (1991) used a modified TAR model to analyze the GDP of six OECD countries, where the threshold variable is the GNP of U.S.. They also found the modified TAR model has substantial improvement in the root mean square error of the estimation. Pope and Yadav (1990) used the TAR-F test and CUSUM test to study the problem of mispricing in financial futures and concluded that the time series of mispricing follow a TAR model.

## 4.6 GARCH Model and $Q^2$ Test

The ARCH-type model of Engle (1982) is an attempt to improve time series forecasting. For a first order AR model:

$$x_t = \gamma x_{t-1} + \epsilon_t , \tag{4.6.1}$$

where $\epsilon_t$ is white noise. The conditional mean of $x_t$ is $\gamma x_{t-1}$ while the unconditional mean is zero. The forecast of the time series is improved by the conditional mean. The conditional variance of the model is still a constant. So similarly, if we have a model with varying conditional variance, the forecast of the time series also can be improved. One of the earlier model along this line is the bilinear model of Granger and Andersen (1978). A simple bilinear model is given by:

$$x_t = \epsilon_t x_{t-1} , \tag{4.6.2}$$

where $\epsilon_t$ is white noise with variance $\sigma^2$. The conditional variance of $x_t$ is $\sigma^2 x_{t-1}^2$. But the unconditional variance of $x_t$ is either zero or infinity, which makes this model unattractive. The alternative model which also has a conditional variance is proposed by Engle (1982). Engle's model, the autoregressive conditional heteroskedasticity (ARCH) model, is defined as:

$$x_t | \Psi_{t-1} \sim N(0,h_t) ,$$
$$h_t = h(x_{t-1}, x_{t-2}, \ldots, x_{t-p}, \alpha) , \qquad (4.6.3)$$

where $\Psi_t$ is the information set available at time t, $\alpha$ is a vector of unknown parameters, p is order of the ARCH process.

The ARCH model was further extended by Bollerslev (1986) to allow for a more flexible lag structure in the variance. A generalized ARCH (GARCH) process of Bollerslev is given by:

$$x_t | \Psi_{t-1} \sim N(0,h_t) ,$$
$$h_t = \alpha_0 + \sum_{i=1,q} \alpha_i x_{t-i} + \sum_{j=1,p} \beta_j h_{t-j} , \qquad (4.6.4)$$

where $p \geq 0$, $q \geq 0$, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $i=1,\ldots,q$, $j=1,\ldots,p$. The GARCH process has an adaptive learning mechanism since it allows lagged conditional variances to determine the current conditional variance.

The GARCH model specified in equation (4.6.4) is denoted by GARCH(q,p). When p=0, the GARCH(q,p) reduces to ARCH(q) process. And when p=q=0, $x_t$ is just simply a white noise. A simple GARCH model is the GARCH(1,1) illustrated by:

$$x_t = \epsilon_t h_t^{1/2} \ , \quad \epsilon_t \sim N(0,1) \ ,$$

$$h_t = \alpha_0 + \alpha \, x_{t-1}^2 + \beta \, h_{t-1} \ , \qquad (4.6.5)$$

where $\alpha_0 > 0$, $\alpha \geq 0$, $\beta \geq 0$ . The GARCH model illustrated above has the property that large (small) changes to be followed by large (small) changes in the time series generally seen in financial markets.

One test of ARCH-type nonlinearity is the Lagrange multiplier test proposed by Engle (1982). After some mathematical derivations and simplifications, Engle showed that the Lagrange multiplier test statistic is asymptotically equivalent to the following statistic:

$$\zeta = T R^2 \ , \qquad (4.6.6)$$

where T is the sample length, $R^2$ is the coefficient of determination of the regression of $x_t^2$ over a constant and its r lagged values. The test statistic $\zeta$ approximates $\chi_r^2$, the chi-square distribution with r degrees of freedom. The null hypothesis of h = constant will be accepted or rejected depend on whether $\zeta$ is within or out side of the acceptance region of the chi-square distribution at the given level of significance.

A similar method by McLeod and Li (1983) also can be used to test the ARCH-type process. McLeod and Li developed a $Q^2$-statistic, following the suggestion of Granger and Andersen (1978) that the autocorrelation function of the square of a time series can be useful in identifying nonlinear bilinear time series. McLeod and Li also suggested the $Q^2$-statistic to be used in identifying other types of nonlinear time series, including the time series of ARCH-type model.

For the time series $\{x_t\}$, its $Q^2$-statistic is given by:

$$Q^2(p)=T(T+2)\Sigma_{k=1,p}\, r^2(k)/(T-k)\ ,$$

$$r(k)=\Sigma_{t=1,T-k}(x_t^2-u^2)(x_{t+k}^2-u^2)/\Sigma_{t=1,T}(x_t^2-u^2)^2\ ,$$

$$u^2=\Sigma_{t=1,T}x_t^2/T\ . \tag{4.6.7}$$

Under the null hypothesis of there is no autocorrelation in the squared values of the time series, the asymptotic distribution of the $Q^2$-statistic will have a $\chi^2$ distribution with p degrees of freedom. However, the exact finite sample distribution of the $Q^2$ statistic can not be obtained analytically, and there has been very few studies of the finite sample behavior of the $Q^2$ test. In Chapter 5 we present a study of the $Q^2$ test.

The estimation of ARCH type model requires maximum likelihood method. For the GARCH model of equation (4.6.4), $x_t$ has a normal distribution with zero mean and variance of $h_t$. So the probability density function (pdf) of $x_t$ is:

$$(2\pi h_t)^{-1/2}\exp(-x_t^2/(2h_t))\ .$$

And the joint pdf for the sample is:

$$\Pi_{t=1,T}(2\pi h_t)^{-1/2}\exp(-x_t^2/(2h_t))\ .$$

Then the log likelihood function is, apart from some constant:

$$L_T(\theta) = T^{-1}\textstyle\sum_{t=1,T} l_t(\theta) ,$$

$$l_t(\theta) = -(1/2) \log h_t - (1/2)\, x_t^2 h_t^{-1} . \tag{4.6.7}$$

where T is the sample length. The Berndt, Hall, Hall and Hausman (1974) algorithm will be used to obtain the maximum likelihood estimates.

Bollerslev (1987) used a GARCH(1,1) model to fit some daily exchange rates and monthly stock prices. Before fitting of the model, Bollerslev used the McLeod-Li $Q^2$ test to show that the data series has ARCH-type nonlinearity. However, after the fitting of the GARCH(1,1) model, the residuals did not show ARCH-type nonlinearity, and Bollerslev concluded that the GARCH(1,1) model fits the series quite well.

Hsieh (1989) studied five foreign currencies, and found the existence of nonlinearity in the data. Hsieh performed several tests, including McLeod-Li $Q^2$ test, and suggested the nonlinearity entered the data through the variance. So Hsieh applied GARCH(1,1) model to the data. The nonlinearity test of standardized residuals of the model showed that the nonlinearity was reduced greatly. Hsieh concluded GARCH(1,1) can account most of nonlinearity of the data. Hsieh (1991) also investigated the returns of several stock indices and found the ARCH-type model can improve the fit to the data.

Heimstra (1990) applied McLeod-Li $Q^2$ test on residual of market models of 270 stocks, and found that nearly 50% of series shows evidence of ARCH type nonlinearity. Yang and Brorsen (1993) studied some agriculture futures, metal futures, and financial futures. They detected nonlinearity in the data. After fitting a GARCH(1,1) model to the data, they found the nonlinearity was removed and the Kurtosis was reduced. They concluded that, while not perfect, the GARCH model was preferable for describing futures prices.

## 4.7 Bispectral Test

Concerned with the inability of autocovariance methods to detect nonlinear serial dependence in time series data[13], Hinich (1982) developed a non-parametric test of nonlinearity using the sample bispectrum. To illustrate, let $\{x_t\}$ denote a third order stationary time series, and let $E[x_t] = 0$ to simplify our discussion. The bispectrum of the time series is defined as:

$$B_x(f_1,f_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{xxx}(m,n)\exp[-i2\pi(f_1 m+f_2 n)] \quad , \qquad (4.7.1)$$

where $f_1$ and $f_2$ are in the domain of the triangular set $A=\{0<f_1<1/2,\ f_2<f_1,\ 2f_1+f_2<1\}$, and the third order cumulative function $c_{xxx}(m,n)$ is given by:

$$c_{xxx}(m,n) = E[x_{t+m}x_{t+n}x_t] \ . \qquad (4.7.2)$$

Because the power spectrum of the time series is:

[13] See Ashley et. al. (1986).

$$S_x(f) = E[|\sum_{n=0}^{\infty} x(n)\exp(-i2\pi fn)|^2] \tag{4.7.3}$$

We define the standardized bispectrum of the time series as follows:

$$\frac{B_x(f_1,f_2)}{[S_x(f_1)S_x(f_2)S_x(f_1+f_2)]^{1/2}} \tag{4.7.4}$$

Hinich and Patterson (1989) showed that the bispectrum of a time series $\{x_t\}$ can be consistently estimated using a sample $\{x_0, x_1, ..., x_{T-1}\}$ as follows. First we calculate the estimate of the bispectrum as:

$$F_x(j,k)=X(j/T)X(k/T)X^*((j+k)/T)\ , \tag{4.7.5}$$

where j and k are integers and

$$X(j/T) = \sum_{t=0}^{T-1} x_t\exp(-i2\pi jt/T)\quad . \tag{4.7.6}$$

X(0) is set to zero because we assume the sample mean is zero. $F_x(j,k)$ is an estimator of the bispectrum of $\{x_t\}$ at frequency pair (j,k). The consistent estimator of the bispectrum is obtained by averaging of $F_x(j,k)$ over adjacent frequency pairs:

$$\overline{B}_x(f_m,f_n) = M^{-2} \sum_{j=(m-1)M}^{mM-1} \sum_{k=(n-1)M}^{nM-1} F_x(j,k)\quad . \tag{4.7.7}$$

$\overline{B}_x(f_m,f_n)$ is the average value of $F_x(j,k)$ over a square of $M^2$ points, $mM-M<j<mM-1$,

$nM-M<k<nM-1$ , and $f_m = (2m-1)M/2T$ . Ashley et. al. (1986) suggested the choice of the smooth integer $M \approx 0.7T^{1/2}$. Similarly, the consistent estimator of power spectrum is given by:

$$\bar{S}_x(f_m) = M^{-1} \sum_{j=(m-1)M}^{mM-1} |X(j/T)|^2 \tag{4.7.8}$$

The standardized bispectrum, apart from a constant factor, is estimated by:

$$\bar{X}_{m,n} = \frac{\bar{B}_x(f_m,f_n)}{(T/M^2)^{1/2}[\bar{S}_x(f_m)\bar{S}_x(f_n)\bar{S}_x(f_{m+n})]^{1/2}} \tag{4.7.9}$$

Hinich (1982) showed that the estimators $2|\bar{X}_{m,n}|^2$ are asymptotically distributed as independent non-central chi-squared, $\chi^2(2,\lambda_{m,n})$, with non-centrality parameter:

$$\lambda_{m,n} = \frac{2M^2|B_x(f_m,f_n)|^2}{T\, S_x(f_m)S_x(f_n)S_x(f_{m+n})} \tag{4.7.10}$$

for all m and n such that the averaging lattice square of $f_m$ and $f_n$ lies entirely within the frequency domain of $A=\{0<f_1<1/2, f_2<f_1, 2f_1+f_2<1\}$. The number of such frequency pairs is denoted by P.

Ashley et. al. (1986) showed that, under the null hypothesis that the time series $\{x_t\}$ is linear, $\lambda_{m,n}$ is a constant, independent of m and n. This constant is consistently estimated by:

$$\lambda_0 = 2\sum_m \sum_n |\bar{X}_{m,n}|^2/P - 2 \tag{4.7.11}$$

If the null hypothesis is true, then the estimators $2|\bar{X}_{m,n}|^2$ will be asymptotically distributed as the $\chi^2(2,\lambda_0)$ distribution, and their sample dispersion will be consistent with the dispersion of $\chi^2(0,\lambda_0)$ distribution. If the null hypothesis is false, so that the time series $\{x_t\}$ is nonlinear, then $\lambda_{m,n}$ will not be a constant. Therefore $2|\bar{X}_{m,n}|^2$ will not be asymptotically distributed as $\chi^2(2,\lambda_0)$ distribution, and their sample dispersion will exceed the dispersion of $\chi^2(0,\lambda_0)$.

One way of measuring the sample dispersion is to compare the 80% quantile of the sample distribution with that of the $\chi^2(2,\lambda_0)$ distribution (Hinich and Patterson, 1988). David (1970) showed that the sample 80% quantile, $\hat{q}_{.8}$, is asymptotically distributed as $N(q_{.8},\sigma_0^2)$, where $\sigma_0^2$ is consistently estimated by:

$$\hat{\sigma}_0^2 = 0.8(1-0.8)f^{-1}(\hat{q}_{.8})P^{-1} \quad . \tag{4.7.12}$$

where $q_{.8}$ is the population 80% quantile of $\chi^2(2,\lambda_0)$, and f(.) is the density function of $\chi^2$. Thus $Z=\hat{q}_{.8}/\sigma_0$ is distributed as N(0,1) under the null hypothesis that the time series $\{x_t\}$ is a realization of a linear process.

Ashley et. al. (1986) studied the power of bispectral test using Monte Carlo experiments. The DGPs they considered are time series of bilinear, nonlinear MA, extended nonlinear MA, threshold AR, nonlinear threshold AR, exponential AR models. Their conclusion was that the bispectral test has power to pick up nonlinearities in the data, except for the nonlinear threshold AR model and the exponential AR models where the power of the test is lower. Because the computation of the Bispetral test is very difficult, the Monte Carlo study in this dissertation does not cover the Bispectral test.

However, we still use it in Chapter 6 for study of futures prices. Hinich and Patterson (1985, 1989) performed bispectral test on time series of 15 stock returns. They concluded that the time series of stock returns are nonlinear, non-Gaussian processes. They also rejected the hypothesis that the time series of stock returns are independent .

## 4.8 Summary

Research in the physical sciences has showed that nonlinear dynamic system can generate deterministic chaos that looks random. These feature are similar to what we have seen in price dynamics in financial markets. The research in the physical sciences further has built foundation from which we can use a single observed variable and learn the properties of underlying complex multi-dimensional dynamic system.

These developments in the physical sciences have thus encouraged the use of nonlinear models and techniques in economics and finance. One example is the over lapping generation model which is a nonlinear deterministic model and can produce chaotic behavior. Another example is the option price model where a tent map can be used for option price determination. Recent studies also proposed to use nonlinear stochastic models in economics and finance.

The techniques in the physical sciences for detecting deterministic chaos are not statistical tests. They are not intended to deal with nonlinear time series process neither. This has motivated the development several statistical tests designed to deal with nonlinear time series process. Three such tests are considered in this dissertation. The

BDS statistic is designed to detect if the time series is IID, but when applied to linear fitted time series it also can be used to detect nonlinear process. The TAR-F test can be used to detect threshold autoregressive type nonlinear process. McLeod-Li $Q^2$ statistic can be used to detect the nonlinearity in the variance, i.e., the autoregressive conditional heteroskedasticity. The finite sample properties of the test statistics are especially important for interpreting the empirical test results, and, however, they have not been thoroughly studied. In Chapter 5 we evaluate the performance of these three tests using Monte Carlo experiments.

The two econometric nonlinear models, the threshold autoregressive model and autoregressive conditional heteroskedasticity model, can be useful in study of price dynamics in financial markets. The first model has nonlinearity in conditional mean and the second model has nonlinearity in conditional variance. In Chapter 6 we apply them to the study of price movements in futures markets.

# CHAPTER 5

# MONTE CARLO ASSESSMENT

## 5.1 Introduction

The three statistical tests discussed in Chapter 4, the BDS, TAR-F, and $Q^2$ tests, are very useful for analyzing economic and finance time series data. However, these tests are relatively new and their finite sample behaviors have not been fully investigated. And their finite sample behaviors are is important for the application of these tests. In this chapter we report results from a Monte Carlo study of the finite sample properties of these tests.

Monte Carlo method is a simulation method for solving problems which are difficult to solve analytically. For the application in this dissertation, the Monte Carlo method is used to study the performance of test statistics. Specifically, the test statistics are computed for a large number of simulated time series samples generated from a data generating process (DGP) which simulates a specific time series process such as an AR(1) process. The mean, the standard deviation, and the rejection frequency of the test statistics are calculated. In each Monte carlo experiment, the sample size and the parameters of the DGP are fixed. To study the performance of a test statistic we usually do many Monte Carlo experiments with different sample size, with different parameter values of the DGP, and with different types of DGP.

When the number of experiments is large, the results of Monte Carlo experiments can be presented comprehensively by the response surface method. For example, the response surface of the rejection frequency of a test statistic is obtained by fitting a model which shows changes of the rejection frequency in response to changes in sample size and to changes in the parameter of the DGP. By using the response surface, we can understand the behavior of the test statistic better than from tables full of test results. The response surfaces also can generalize the result of Monte Carlo experiment to other sample sizes and to other parameter values of the DGP. Thus we can learn the properties of the test statistic at new parameter values of the DGP and at new sample sizes without re-do the Monte Carlo experiment. By using results from many Monte Carlo experiments, the response surfaces can also reduce the effect of impreciseness resulted from individual experiments.

Underlying all statistic investigations is the concept of random experiment. The random experiment is even more essential for Monte Carlo simulation. An important procedure in the Monte Carlo simulation is the generation of random numbers and the time series samples which will be used in the simulation. A good random number generator is necessary for the success of the Monte Carlo simulation. The commonly used method for generating random numbers is the Multiplicative Congruential Generator which generates random numbers with uniform distribution. The random numbers with other distributions and the simulated time series samples of different DGPs can be generated from random numbers with uniform distribution.

In this chapter, we review Monte Carlo experiment and the response surface method in Section 5.2. In Section 5.3 we discuss the data generating process (including the random generator) and the Monte Carlo experiment design. The results of Monte Carlo studies of the BDS, TAR-F, and $Q^2$ test are presented in Sections 5.4, 5.5, and 5.6 respectively. Section 5.7 compares the results of different test statistics. Finally Section 5.8 summarizes the results of Monte Carlo study.

## 5.2 Monte Carlo Experiment and Response Surface Method

Monte Carlo method is a special method of simulation. It gets its name from the gambling city in France. During the World War II, when scientists in U.S. were studying the fission process of the atomic bomb, they developed a numerical simulation method, code named Monte Carlo for the purpose of war-time secrecy.[1] Later on Monte Carlo method has been applied to many other fields for solving the problems which are too difficult to solve analytically. The application of Monte Carlo method to the study of finite sample properties of statistical estimation is relatively new. In this dissertation we use Monte Carlo method to study the finite sample properties of three statistical tests.

To illustrate, suppose we have an estimator (or a test statistic), and we want to know the properties of this estimator. When applying the estimator to empirical data, we need to know the finite sample properties of the estimator for interpreting the results of

[1] See Rubinstein (1981) for different application of Monte Carlo method.

the estimation. For most of the estimators, we know, at most, the asymptotic properties of the estimator. And usually the finite sample properties of the estimator are difficult to derive analytically, so we rely on Monte Carlo method to study the finite sample properties of the estimator.

Specifically, if we have an estimator relating to a time series model, the finite sample properties of the estimator at which we are interested are the mean and the standard deviation. In a Monte Carlo experiment, we generate many simulated time series samples from a data generating process (DGP) which represents the time series model. In each experiment, the sample size and parameter value of the DGP are fixed. The estimator is applied to these samples. Then the mean and the standard deviation of the estimator can be estimated from replicated calculation of the estimator based on these samples. For a test statistic, we also estimate the rejection frequency of the test from replicated statistics on these samples.

Suppose $\eta$ is a test statistic, for given sample size T and for given DGP parameter $\phi$, $\eta_i$ is the calculation of $\eta$ from each simulated sample. Then the Monte Carlo estimates of the mean, of the standard deviation, and of the rejection frequency of the test statistic can be obtained from:

$$h = \Sigma\eta_i/N ,$$

$$s = \Sigma(\eta_i-h)^2/(N-1) ,$$

$$P = \Sigma I_i/N , \qquad (5.2.1)$$

where N is the number of replications, $I_i = 1$ if and only if $\eta_i \geq d_1$ and is zero otherwise, and $d_1$ is the percentile value of the test statistic.

The quantities of interest such as those in equation (5.2.1) are themselves estimates from Monte Carlo experiment, and therefore are subject to experimental error. This error can be reduced acceptably small by using a sufficiently large number of replications and perhaps by using some variance reduction techniques. The two important variance reduction techniques are antithetic variance and control variate. In the antithetic variance method, two estimates are obtained for one Monte Carlo experiment, and the pooled estimate is used. When covariance of the two estimates is negative, the variance of the pooled estimate can be reduced below the ordinary estimate. A control variate is an random variable of which the distribution is know and that is correlated to the quantity of interest. And the divergence between the sample mean of the control variate in the experiment and its known population mean is used to improve the estimate from the Monte Carlo experiment.[2]

For each set of sample size and parameter of DGP, $\{T,\phi\}$, equation (5.2.1) gives us the results of one Monte Carlo experiment. If we change sample size and the parameter of the DGP, the result of the Monte Carlo experiment will also change accordingly. And in general, for a Monte Carlo study of a test statistic, we will do many Monte Carlo experiments with different sample sizes and with different values of the

[2] The design and implementation of the antithetic variance and control variate are often difficult, so in this dissertation we will rely on using large number of replications for variance reduction of the Monte Carlo estimates. For detailed discussion and simple examples, see Davidson and MacKinnon (1993).

DGP parameter. Then we analyze how changes in sample size and parameter value of DGP affect the test statistic. When the number of Monte Carlo experiments is small, we can present the results in tables. But when the number of Monte Carlo experiments is large, as the case in this dissertation, presenting results in pages of tables can make the results difficult to understand. In this case, the method of response surface becomes a better way to present the results of Monte Carlo experiments.

The response surface is simply a regression model in which each observation correspond to one experiment, the dependent variable is the quantity of interest that was estimated in the experiment, and the independent variables are functions of sample size and of the parameters of the DGP. From a response surface we can understand the behavior of the quantity of interest easier than from pages of tables full of numbers. The response surface also reduces the problem of specificity, provides results for a range of sample sizes and of parameter values of DGP rather than for just the sample sizes and parameter values chosen by experimenter, and eliminates the need to repeat the experiment at every new sample size and every new parameter value of the DGP (see Hendry, 1984). When the number of experiments is large, the response surface can be estimated with great precision even when the number of replications is small, because the large number of experiments can compensate for imprecise results from each individual experiment.[3] A major criticism of the response surface method is that the regression model of the response surface can be misspecified and thus can give us false

[3] See discussion in Hendry (1984), Davidson and MacKinnon (1993). For example, Engel, Hendry, and Trumble (1985) use only 21 replications per experiment in the response surface regression.

representation of the behavior of the quantity of interest.[4] However, if we take extra caution in specifying the regression model for the response surface, we can reduce the problem of misspecification. Furthermore, if we plot the results of individual experiment along with the estimated response surface, we can check whether the response surface obtained is too far away from the actual results of the Monte Carlo experiments.

To demonstrate, denote the quantity of interest by g. It is a function of the sample size and of parameters of DGP, which we denote by the vector $\phi$. We model this function by $G(T,\phi,\gamma)$, where G is a specific functional form that depends on the parameter vector $\gamma$, which will be estimated. This model tells us how g responds to changes on T and $\phi$. Denote $g_j$ as the Monte Carlo estimate of g from j-th experiment, which has a estimated standard error $\sigma(g_j)$. We assume the number of replication per experiment is large so we can be confident that $g_j$ is very close to being normally distributed with mean $G(T,\phi,\gamma)$ and standard deviation $\sigma(g_j)$ which can be well estimated by $\sigma(g_j)$. Thus we can write response surface as:

$$g_j = G(T,\phi,\gamma) + v_j \,,\ v_j \sim N(0,\sigma^2(g_j)) \,,\ j=1,\ldots,n, \tag{5.2.2}$$

where n is the number of experiments and hence the number of observation for the response surface regression. To eliminate heteroskedasticity, we do the following transformation:

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[4] See Maasoumi and Phillips (1982), and the reply by Hendry (1982).

$$g_j/\sigma(g_j) = G(T,\phi,\gamma)/\sigma(g_j) + u_j\ ,\ u_j \sim N(0,1)\ ,\ j=1,\ldots,n. \qquad (5.2.3)$$

For a test statistic, let $h_j$, $s_j$, and $P_j$ the estimated mean, standard deviation, and rejection of test statistic obtained from j-th experiment based on equation (5.2.1). For the mean of the test statistic, we can do the response surface regression by using $h_j$ directly, and the associated standard error is $1/\xi_1=s_j/N^{1/2}$. For the standard deviation of the test statistic, we need a logarithm formulation to ensure the positive prediction of s from the regression of response surface. So $\log(s_j)$ is used in the regression and the associated standard error is $[var(s_j)]^{1/2}/(2N^{1/2}s_j^2)$. If $\eta_i$ is normally distributed, then $var(s_j)=2s_j^4$. Thus the associated standard error will be $1/\xi_2=1/(2N)^{1/2}$.[5]

For the rejection frequency of the test statistic, we need a logit transformation to ensure the predicted rejection frequency is between 0 and 1 from the regression of response surface, that is $L(P)=\log[P/(1-P)]$. Thus for $L(P_j)$, the associated standard error is $1/\xi_3=[NP_j(1-P_j)]^{1/2}$.[6] After the logit transformation, the observations with $P_j=0$ or $P_j=1$ are omitted from the regression of response surface.

To specify the regression model for the response surface, we consider the following. For most of test statistics, their asymptotic distributions under the null hypothesis (ADUNH) are generally known. So when studying the finite sample distribution (FSD) of the test statistic, we examine the deviation of FSD from ADUNH.

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[5] See the discussion in Hendry (1984). Here I consider the standard deviation rather than the variance of the test statistic, but the derivation of the standard error of the regression model should be same.

[6] See Hendry (1984).

Under the null hypothesis, the deviations from ADUNH will be large if the sample size is small and the deviation will be small if the sample size is large. Under the alternative hypothesis, we formulate the DGPs so that the parameters of the DGPs signify the departure from the null hypothesis. If the parameters are zero, the DGPs of alternative hypothesis reduce to the DGP of the null hypothesis. Thus the deviations of FSD from ADUNH will be large if the parameter of the DGP is large and the deviation will be small if the parameter of the DGP is small. Furthermore, under the alternative hypothesis, the deviation of FSD from ADUNH will be large if the sample size is large because the test will reject the null with certainty when the sample size is infinitely large. Thus we consider the terms of $1/T$, $1/T^2$, $\phi$, $\phi/T$, $\phi/T^2$, $\phi^2$, $\phi T$, $\phi T^2$ plus a constant term in the general regression model of response surface and then simplify the model by restricting some coefficients to equal zero.[7]

For the BDS statistic, we study its finite sample distribution under both the null and alternative hypothesis. The mean of the BDS statistic from ADUNH is zero, and the standard deviation of the BDS statistic from ADUNH is 1, the regressions:

$$h_j\xi_1=(a_{10}+a_{11}/T+a_{12}/T^2+a_{13}\phi+a_{14}\phi/T+a_{15}\phi/T^2+a_{16}\phi^2+a_{17}\phi T+a_{18}\phi T^2)\xi_1+e^h_j ,$$

$$\begin{aligned}(\log(s_j))\xi_2=(&a_{20}+a_{21}/T+a_{22}/T^2+a_{23}\phi+a_{24}\phi/T+a_{25}\phi/T^2\\ &+a_{26}\phi^2+a_{27}\phi T+a_{28}\phi T^2)\xi_2+e^s_j ,\end{aligned} \tag{5.2.4}$$

tell us the behaviors of the mean and the standard deviation of the BDS statistic in

[7] See Davidson and MacKinnon (1993).

response to variations of sample size and parameters of DGP. The terms -0 and -log(1) are dropped from the left hand side of the above equations because they are zero and have not effect in the equation.

For the BDS, TAR-F, and $Q^2$ test, we study the behavior of their rejection frequency. For fixed percentile value under the null hypothesis, we analyze the departure of the rejection frequency from its value of ADUNH, $\delta$. Thus we estimate the regression model:

$$(L(P_j)-L(\delta))\xi_3=(a_{30}+a_{31}/T+a_{32}/T^2+a_{33}\phi+a_{34}\phi/T+a_{35}\phi/T^2$$
$$+a_{36}\phi^2+a_{37}\phi T+a_{38}\phi T^2)\xi_3+e^p{}_j, \tag{5.2.5}$$

which shows the variation of rejection frequency in respond to variations in sample size and to variation in the parameter of the DGP.

In this dissertation, I do not attempt to analyze the size power trade-off of these tests. To analyze the size and power trade-off, we need to look the rejection frequency at many percentile values for each experiment. Because of the large number of experiments involved in this dissertation and of space limitation, we only analyze the rejection frequency a one percentile value. Even if we consider only one percentile value, we can still obtain useful results for the application of the tests to empirical data.

## 5.3 Data Generating Process and Experimentation Design

In order to carry out a Monte Carlo study, we need to generate simulated samples which will be used for the experiments. And random numbers are needed for generating simulated samples. To generate random numbers, we first with generate random numbers with uniform distribution and then derive random numbers with other distribution from the random numbers with uniform distribution.

Like most other Monte Carlo studies, the random numbers used in this dissertation will be generated by using of digital computer. However, no digital computer is able to generate genuine random numbers. Digital computers can only generate sequence of pseudo-random numbers, which are in fact deterministic. A good random number generator in computer program can generate pseudo-random numbers that are indistinguishable from genuine random numbers for the purpose of Monte Carlo experiment. One frequently used random number generator is the Multiplicative Congruential Generator:

$$n_t = z_t/r, \text{ with } z_{t+1} = b\, z_t \ (\text{mod } r)\ ,\ t=0,\ 1,\ 2,\ \dots\ , \tag{5.3.1}$$

which produces pseudo-random numbers with uniform distribution in the interval (0,1). The choices of b and r are important for the random number to be uncorrelated and uniform. In general, we chose $b = 7^5$, and $r = 2^{31} - 1$ .

Marsaglia and Bray (1968) developed a shuffling method to decrease any dependency in the pseudo-random number generating process. Before the pseudo-random numbers are generated, we first create a table of 128 uniform distributed random numbers. Then for each new pseudo-random number, two seed values are used. The number generated from the first seed is used to pick a position out of 128 numbers in the table. The number in that position is used as the new pseudo-random number, and is replaced by the number generated from the second seed. This process will be continued until the desired amount of pseudo-random numbers is generated.

In most econometric models, we assume the error term has normal distribution. And the error term in the DGP used for most Monte Carlo experiment is also assumed to be normally distributed. A popular technique for generating random numbers with normal distribution is the Box-Muller bivariate method:

$$\hat{}(e^1_t, e^2_t) = (-2\ln(n_{2t-1}))^{1/2} (\cos 2\pi n_{2t}, \sin 2\pi n_{2t}) . \qquad (5.3.2)$$

where $n_t$'s are random numbers with uniform distribution, and the random numbers generated, $e_t$'s, have standard normal distribution N(0,1). Thus for samples from DGP of time series model with serial dependence, we can pick an initial value for the time series and do recursive calculation based on the DGP. To reduce the sample's dependence on the initial value of the time series, we generate additional 200 observations on top of the given sample size and then discard the first 200 observations.

The design of the Monte Carlo experiments in this dissertation is based on the following consideration. First, we select 9 types of DGP which represent 9 types of time series model. They are 3 IID time series models, 2 linear time series models, and 4 nonlinear time series models. The 3 IID time series models fall the under null hypothesis of the BDS statistic and under the null hypothesis of the $Q^2$-statistic. The 2 linear time series models and the 4 nonlinear time series models fall under the alternative hypothesis of the BDS statistic and under the alternative hypothesis of the $Q^2$-statistic. The 3 IID time series models and the 2 linear time series models fall the under null hypothesis of the TAR-F statistic, with remaining 4 nonlinear time series models fall under the alternative of the TAR-F statistic. In this way, we have balanced models under null hypothesis and under alternative hypothesis for each test statistic.

Second, for each DGP of time series model, we keep the number of parameters to the minimum while preserving the basic feature of the model. Therefore we can avoid the difficulty of presenting the results where the effect of many parameters in the DGP has to be considered.

Third, we chose the parameter values of the DGP from zero to modest values so we will not be overwhelmed by the results which have 100% rejection frequency. After all, the observations with 100% rejection frequency are excluded from and do not contribute to the regression of response surface of the rejection frequency.

Fourth, the sample sizes we considered are: T=100, 200, 500, and 1000. This is based on the fact that most time series data used in economic and financial research has observations in this range. For example, one year's daily stock returns have about 250 observations, ten year's weekly economic indicators has about 520 observations.

Fifth, the numbers of replications are: N=1000 for T=100, N=500 for T=200, N=200 for T=500, and N=100 for T=1000. We use more replications for small sample size because for a given number of replications the Monte Carlo estimate of the quantity of interest is less precise with small sample size. The large number of replications will compensate the imprecise brought by small sample size. The smallest number of replication is N=100 for T=1000. With this number of replication, if the size of the test under null hypothesis is 5%, the standard deviation of the size of the test will be:

$$[P(1\text{-}P)/N]^{1/2} = (0.05*0.95/100)^{1/2} = 0.022 \ ,$$

which is acceptably small. Furthermore, the use of response surface with large number of experiments will reduce the effect of impreciseness brought by the individual experiment.

The DGPs and their parameter values used for Monte Carlo studies in the following sections are summarized in Table 5.1. Following Davidson and MacKinnon (1993), if a test statistic is not sensitive to a parameter of a DGP, we use less number of values for that parameter in the Monte Carlo experiments. Therefor, for a test statistic we may use less number of parameter values than those specified in Table 5.1.

Table 5.1

DGPs and Their Parameter Values in Monte Carlo Experiments

**IID Time Series**

a). IID with normal distribution: $x_t \sim N(0,1)$;

b). IID with uniform distribution: $x_t \sim U(0,1)$;

c). IID of bimodal mixture of normals, $x_t \sim \{0.5\ N(0,1) + 0.5\ N(\alpha,\beta^2)\}$, with $\alpha$=0, 1, 2, 4, 6, and $\beta$=1, 2, 4;

**Linear Time Series**

d). Linear AR(1) process: $x_t = \alpha\, x_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$, with $\alpha$=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8;

e). Linear MA(1) process: $x_t = \alpha\, \epsilon_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$, with $\alpha$=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0;

**Nonlinear Time Series**

f). Nonlinear AR(1) process: $x_t = \alpha\, x_{t-1}(1-x_{t-1})/(1+x_{t-1}^2) + \epsilon_t$, $\epsilon_t \sim N(0,1)$, with $\alpha$=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95;

g). Nonlinear MA(1) process: $x_t = \alpha\, \epsilon_{t-1}\, \epsilon_{t-2} + \epsilon_t$, $\epsilon_t \sim N(0,1)$, with $\alpha$=0, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0;

h). Threshold autoregressive process:

$$x_t = \alpha\, x_{t-1} + \epsilon_t \quad \text{if } x_{t-r} < 0 ,$$
$$= -\alpha\, x_{t-1} + \epsilon_t \quad \text{if } x_{t-r} > 0 , \quad \epsilon_t \sim N(0,1) ,$$

with AR coefficient $\alpha$=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, and threshold lag r=1, 2, 3, 4;

i). GARCH(1,1) process:

$$x_t = h_t\, \epsilon_t , \quad \epsilon_t \sim N(0,1) ,$$
$$h^2_t = 1 + \alpha\, x^2_{t-1} + \beta\, h^2_{t-1} ,$$

with $\alpha$=0.05, 0.1, 0.2, 0.3, 0.4, and $\beta$=0.05, 0.1, 0.2, 0.3, 0.4.

## 5.4 Monte Carlo Study of the BDS Statistic

As discussed in Chapter 4, we can use different values of correlation length and different embedding dimensions to calculate the BDS statistic. But then which correlation length and which embedding dimension should be used to calculate the BDS statistic and to make statistic inference?

The most recent and comprehensive study of finite sample properties of the BDS statistic is given by Brock, Hsieh, and LeBaron (1991). Although the authors presented pages of tables full of numbers from Monte Carlo experiments, their number of experiments is still small. For each type of non-IID time series, only one parameter value of the time series is considered. The parameter value considered by them is usually large so the rejection frequency is large. Therefore we still have no knowledge on the performance of the BDS statistic when the parameter value of the non-IID time series is small.

The number of sample sizes and the number of embedding dimensions considered by Brock, Hsieh, and LeBaron are also limited. So when studying the selection of correlation length and of embedding dimension, results of these authors were based on limited cases in which the sample size and embedding dimension take limited values. The authors only give a range of values for the selection of the embedding dimension and correlation length.

In this section we extend the Monte Carlo study of the finite sample properties of the BDS statistic, consider more parameter values of non-IID time series, more number of sample sizes, and more number of embedding dimensions. Then we study the choice of correlation length and embedding dimension for calculating the BDS statistic and for making statistical inference. At the selected correlation length and the embedding dimension we report the results of Monte Carlo experiments using response surfaces to show the effect of sample size and the effect of parameter of the time series on the performance of the BDS statistic.

### 5.4.1 Selecting Embedding Dimension and Correlation Length

The calculation of the BDS statistic is given by equation (4.4.4) and the related equations. The BDS statistic calculated then will depend on, aside from the parameters of the time series and sample size, the correlation length and the embedding dimension. The correlation length is the measure of how close will the two points be considered as "correlated", and it has the same unit as the time series data. To make it into a "standardized" measure, the correlation length is generally put into the unit of the sample standard deviation. This measure is better than the actual unit of the time series or the unit of spread of the time series, because the measure in the unit of the sample standard deviation takes account every data in the sample and is related to the "closeness" or the dispersion of the sample. So after now on when the correlation length is mentioned, it is always in the unit of sample standard deviation.

Many time series data we are concerned in economics and finance are one-dimensional data. But the underlying system of these data may be multi-dimensional.

To put the data into a higher dimensional space can help us to analyze the data and the underlying system. If the embedding dimension in the calculation of the BDS statistic is too small, we will have difficulty to untangle the complex data. However, if the embedding dimension is too large, the non-overlapping pairs of the sample time series in the correlation function will be too few to give us meaningful statistic. So the embedding dimension can not be too small nor too large. Similar consideration also has to be given to the correlation length. If the correlation length is too small, very few pairs of the sample time series will be considered as "correlated". If the correlation length is too large, most pairs of the sample time series will be considered as "correlated". Either way the BDS statistic calculated can have large standard error. So the correlation length should not be too small nor too large.

For the selection of correlation length L and the embedding dimension M, the theoretical works did not give us any value for L nor M. Brock, Hsieh, and LeBaron (1991) did some Monte Carlo investigation and suggested the selection of $0.5<L<1.5$ and $2<M<5$ for small sample (less than 2000 points). Their selection is based on: a) the "good" approximation of finite sample distribution of the BDS statistic of IID time series to the asymptotic distribution which is measured at the selected percentile values; and b) the size and the power of the BDS statistic. But for the non-IID time series, with the parameter of the time series fixed at large values, the power of the BDS statistic at most sample sizes is unanimously 100% regardless the value of L and M. And in this case one can not really tell which value of L and/or M gives larger power on rejecting the null. Their study also has few number of sample sizes and few number of embedding dimensions. Therefore there is need to study the selection of L and M for both IID and non-IID time series using more number of sample sizes, more number of embedding dimensions, more parameter values of time series.

In this section, we study the selection of correlation length L and the embedding dimension M based on the rejection frequency and the standard deviation of the BDS statistic. When the BDS statistics are calculated at various correlation lengths L and embedding dimensions M, the values of L and M which give lowest rejection frequency for IID time series, highest rejection frequency for non-IID time series, and the lowest standard deviation for replicated the BDS statistics will be selected as the optimum choice of L and M. The correlation lengths we considered are L=0.5, 1.0, 1.5, 2.0, and 2.5. And in a few instances we also considered L=0.2 and 3.0. The embedding dimension we considered are M=2, 3, 5, and 7. The sample sizes used in Monte Carlo studies are T=100, 200, 500, and 1000. The types of time series and the parameter values of the time series are specified in Table 5.1.

After the calculation of the rejection frequency and the standard deviation of the BDS statistic, we regress them over L, $L^2$, M, $M^2$, LM, and other terms relate to the sample size and the parameter of the time series. The regression models are:

$$(\log(s_j))\xi_2=(b_{21}L+b_{22}L^2+b_{23}M+b_{24}M^2+b_{25}LM+\phi_s(T,\gamma))\xi_2+e^s_j \quad , \qquad (5.4.1)$$

$$(L(P_j)-L(\delta))\xi_3=(b_{31}L+b_{32}L^2+b_{33}M+b_{34}M^2+b_{35}LM+\phi_p(T,\gamma))\xi_3+e^p_j \quad , \qquad (5.4.2)$$

where $\phi_s(T,\gamma)$ and $\phi_p(T,\gamma)$ represent other terms on right hand side of equations (5.2.4) and (5.2.5) respectively.

After obtaining the estimated coefficients of L, $L^2$, M, $M^2$, and LM, we can use them to get the minimum or maximum points of the rejection frequency and the minimum points of the standard deviation of the BDS statistic. The estimated coefficients are presented in Table 5.2 along with the optimum values of L and M for rejection frequency or standard deviation.

From Table 5.2 we can see that for IID time series of normal distribution, the small embedding dimension M will give small rejection frequency and small standard deviation of the BDS statistic (see Figure 5.1). When M=3, then minimizing point of correlation length for rejection frequency is at L=1.52 and for standard deviation is at L=1.54.

For IID time series with uniform distribution, to get small rejection frequency of the BDS statistic we need to have small M. When M=3, we can see that L=1.81 will minimize the rejection frequency. The embedding dimension is preferably small to get small standard deviation of the BDS statistic. If we select M=3, then the minimizing point of L is 1.79 for standard deviation of the BDS statistic.

For IID time series with bimodal normal distribution, the small embedding dimension M will also give small rejection frequency and small standard deviation of the BDS statistic. At M=3, the rejection frequency of the BDS statistic will be minimized at L=1.62, and the standard deviation of the BDS statistic will be minimized at L=1.72.

For time series of linear AR(1) process, the rejection frequency of the BDS will be large if the embedding dimension M is small, and it will be maximized when the correlation length L=1.63 (see Figure 5.2). The small embedding dimension M will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when L=1.70.

For time series of linear MA(1) process, the rejection frequency of the BDS statistic will be large if the embedding dimension is small, and it will be maximized when correlation dimension L=1.46. The small embedding dimension will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when L=1.47.

For time series of nonlinear AR(1) process, the small values of embedding dimension M and/or small values of correlation length L will give large rejection frequency of the BDS statistic (see Figure 5.3). The small embedding dimension M will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when L=1.64.

For time series of nonlinear MA(1) process, the rejection frequency of the BDS statistic will be maximized when embedding dimension M=4.68 and correlation length L=1.47. The small embedding dimension will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when correlation length L=1.45.

For time series of TAR process, the small values of embedding dimension M and/or small values of correlation length L will give large rejection frequency of the BDS statistic. This is very similar to the case for time series of nonlinear AR(1) process. The small embedding dimension M will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when L=1.57.

For time series of GARCH(1,1) process, the rejection frequency of the BDS statistic will be maximized when embedding dimension M=2.10 and correlation length L=2.12. The small embedding dimension M will give small standard deviation of the BDS statistic. At M=3, the standard deviation of the BDS statistic will be minimized when correlation length L=1.59.

To summarize, we have seen that for IID time series, the rejection frequency of the BDS statistic will be small if the embedding dimension is small, and at M=3, the rejection frequency will be minimized when the correlation length is in the range of 1.5 to 1.8. In general, for non-IID time series, the rejection frequency of the BDS statistic will be large if the embedding dimension M is small. For nonlinear MA(1) and GARCH(1,1) time series, the regression results indicated that the rejection frequency of the BDS statistic will be maximized at M=4.68 and M=2.10, respectively. And for most of non-IID time series, the rejection frequency of the BDS statistic will be maximized when correlation length takes value from 1.39 to 1.64. These values are independent of embedding dimension or are obtained when embedding dimension is set at M=3. For nonlinear AR(1) and TAR process, the rejection frequency of the BDS statistic will be large if the correlation length is small.

For all the time series studied, the standard deviation of the BDS statistic will be small if the embedding dimension M is small. If we set M=3, the standard deviation of the BDS statistic will be minimized when correlation length L is in the range of 1.3 to 1.8. Overall, the dependence of the rejection frequency and of standard deviation of the BDS statistic on embedding dimension M is weak compared to their dependence on correlation length L.

Therefore if we want to chose a single embedding dimension M and a single correlation length L to simplify the calculation of the BDS statistic for time series we studied, the choice of M=3 and L=1.5 comes out as a strong candidate. This is consistent with the suggestion of selecting $0.5<L<1.5$ and $2<M<5$ by Brock, Hsieh, and LeBaron (1991). The choice of single M=3 and L=1.5 given in this section can help us to reduce the work of calculating the BDS statistic at many other embedding dimensions and many other correlation lengths. We also get better results with this choice than with other choice of embedding dimension and correlation length for most of the time series models considered in this section.

### 5.4.2 Response Surfaces

In this sub-section, the BDS statistics of various time series with different parameter values and different sample sizes are calculated at embedding dimension M=3 and correlation length L=1.5. The Monte Carlo estimate of the means, the standard deviations, and the rejection frequencies of the BDS statistic are regressed according to the general models specified in equations (5.2.4) and (5.2.5). Following Davidson and MacKinnon (1993), we simplify the general models by restricting some coefficients, whose t-statistics are low, to zero. The final regression results are presented in Table 5.3. To help us visualize the response surfaces, the response surfaces are also depicted in the Figures 5.4 through 5.12, where the markers are the results of individual experiments and the lines are from the regression models of response surfaces.

Under the null hypothesis that the time series is IID, the BDS statistic is asymptotically standard normal distributed. In this case the mean of the BDS statistic should be zero, the standard deviation of the BDS statistic should be one, and the rejection frequency of the BDS statistic at 1.96 should be 2.5%. We use these as the bench marks in our investigation of the finite sample properties of the BDS statistic on different types of time series. The following are the results of our Monte Carlo experiments.

For IID time series with N(0,1) normal distribution, the mean of the BDS statistic tend to be negative (see Figure 5.4). But as the sample size increases, the mean will move toward zero. The standard deviation of the BDS statistic is larger than one. And as the sample size increases, the standard deviation will move toward one. The rejection frequency of the BDS statistic is larger than 2.5%, but will move toward 2.5% as the sample size increases.

For IID time series with U(0,1) uniform distribution, the mean of the BDS statistic is negative for small sample size, but is greater than -0.14 (see Figure 5.5). When the sample size is large, the mean will increase to some positive number, about 0.03. As the sample size increases, the standard deviation of the BDS statistic will decrease from larger than one to about one. When the sample size is small, the rejection frequency is about 7%. The rejection frequency will decrease to 4% when the sample size is large.

For IID time series with bimodal normal distribution, where $\alpha$ and $\beta$ are respectively the mean and the standard deviation of the second mode of the normal distribution, the increase of $\alpha$ will lower the mean of the BDS statistic (see Figure 5.6). The increase of $\beta$ will lower the mean of the BDS statistic when sample size is small, but will raise the mean of the BDS statistic when the sample size is large. In general the increase of the sample size will raise the mean of the BDS statistic. But when $\alpha$ is large and $\beta$ is small, the increase of the sample size will lower the mean of the BDS statistic. The increase of $\alpha$ and the increase of $\beta$ can raise or lower the standard deviation of the BDS statistic, depending their values and the sample size. The increase of the sample size will lower the standard deviation of the BDS statistic. The increase of $\alpha$, the increase of $\beta$, and the sample size have mixed effect on the rejection frequency of the BDS statistic. The increase of $\alpha$ will raise the rejection frequency of the BDS statistic if $\alpha$ is large and $\beta$ is small. The increase of the sample size will lower the rejection frequency of the BDS statistic when $\beta$ is small. But when $\beta$ is large and the sample size is large, the increase of $\alpha$ will lower the rejection frequency of the BDS statistic. When $\alpha$ is small and $\beta$ is large, the increase of the sample size will raise the rejection frequency of the BDS statistic. Overall, the mean, the standard deviation, and the rejection frequency of the BDS statistic have weak dependence on $\alpha$ and $\beta$, and their values do not depart far from the bench mark values compared to their behavior in the case of non-IID time series.

For linear AR(1) process, the mean of the BDS statistic will increase when the AR coefficient increases and/or the sample size increases (see Figure 5.7). The standard deviation of the BDS statistic will increase when the AR coefficient increases, but with no strong dependence on the sample size. The rejection frequency of the BDS statistic will increase when AR coefficient increases and/or sample size increases. When the AR coefficient $\alpha$=0.3 and the sample size T increases from 100 to 1000, the rejection frequency will increase from 30% to nearly 100%. At the sample size of T=100, when the AR coefficient increases from 0.1 to 0.6, the rejection frequency will increase from 8% to nearly 100%.

For the linear MA(1) process, the nonlinear AR(1) process, and nonlinear MA process, the behaviors of the mean, the standard deviation, and the rejection frequency of the BDS statistic are similar to the case of linear AR(1) process (see Figures 5.8, 5.9, and 5.10). The only differences are when the parameter of the time series or the sample size changes, the means, the standard deviation, and the rejection frequency of the BDS statistic will not change by the same magnitude compared to their changes in the case of the linear AR process.

For threshold autoregressive process, the mean of the BDS statistic will increase as the AR coefficient increases, and/or the threshold lag increases, and/or the sample size increases (see Figure 5.11). The standard deviation of the BDS statistic will increase as the AR coefficient increases, and/or the threshold lag increases, and/or the sample size increases. But the dependence of the standard deviation on the threshold lag is not strong. The rejection frequency of the BDS statistic will increase when the

AR coefficient increases, and/or the threshold lag increases, and/or the sample size increases.

For time series of GARCH(1,1) process, the mean of the BDS statistic will increase if either one or both GARCH(1,1) coefficients increase, and/or the sample size increase (see Figure 5.12). The standard deviation of the BDS statistic will increase if the first coefficient of GARCH(1,1) increases and/or the sample size increases. The second coefficient of GARCH(1,1) has mixed effect on the standard deviation of the BDS statistic, its increase can raise or lower the standard deviation of the BDS statistic, depending on the values of the first coefficient of GARCH(1,1) and the sample size. The rejection frequency of the BDS statistic will increase if either one or both GARCH(1,1) coefficients increase and/or the sample size increase.

In summary, the results show that the means, the standard deviation, and the rejection frequency of the BDS statistic of IID time series will move close to their asymptotic values under null hypothesis as the sample size increases. For the IID time series of uniform and bimodal mixture of normal, the distributions of the BDS statistic are biased from the standard normal under null hypothesis even when the sample size is very large. But their departure from the standard normal is small in comparison to that found in case of non-IID time series. we also found that for non-IID time series, as the parameter of the time series increases (representing the departure from an IID time series), the mean and the standard deviation of the BDS statistic will increase to larger values from their respectively values of zero and one under null hypothesis. Furthermore, the rejection frequency of the BDS statistic approaches unity as the parameter value increases. In general, as the sample size increases, the mean, the standard deviation, and the rejection frequency of the BDS statistic will increase. When the parameter values of non-IID time series are small, and/or sample size is small, the distribution of the BDS statistic of non-IID time series will be close to the asymptotical distribution under the null hypothesis.

The response surface models obtained in this section are simplified models of the general response surface models given in Section 5.2. Before obtaining these response surface models, we also used more regressor in the models, but found that these models are insensitive to the addition of more aggressor. Further more, the plot of the results from individual Monte Carlo experiments is used for specifying the functional form of the response surface models. Therefore the response surface models obtained do not have major problem of misspecification, and they are robust in terms of insensitive to addition of more regressors.

## Table 5.2
## Selection of Correlation Length L and Embedding Dimension M

| | L | $L^2$ | M | $M^2$ | LM | L* | M* | L*(M=3) |
|---|---|---|---|---|---|---|---|---|
| **IID Time Series with N(0,1) Normal Distribution** | | | | | | | | |
| L(P) | -2.505 | 0.958 | 0.276 | | -0.136 | | small | 1.521 |
| | (0.109) | (0.033) | (0.014) | | (0.009) | | | |
| | n=240 | Adjusted $R^2$=0.892 | | | RSE=0.208 | | | |
| log(s) | -1.200 | 0.489 | 0.207 | | -0.103 | | small | 1.543 |
| | (0.083) | (0.025) | (0.013) | | (0.008) | | | |
| | n=240 | Adjusted $R^2$=0.794 | | | RSE=0.162 | | | |
| **IID Time Series with U(0,1) Uniform Distribution** | | | | | | | | |
| L(P) | -3.615 | 1.040 | 0.113 | | -0.049 | | small | 1.810 |
| | (0.145) | (0.045) | (0.018) | | (0.013) | | | |
| | n=240 | Adjusted $R^2$=0.871 | | | RSE=0.284 | | | |
| log(s) | -3.388 | 1.062 | 0.294 | | -0.139 | | small | 1.792 |
| | (0.085) | (0.025) | (0.013) | | (0.008) | | | |
| | n=240 | Adjusted $R^2$=0.957 | | | RSE=0.165 | | | |
| **IID Time Series with Bimodal Normal Distribution** | | | | | | | | |
| L(P) | -1.469 | 0.565 | 0.249 | | -0.121 | | small | 1.621 |
| | (0.108) | (0.032) | (0.014) | | (0.009) | | | |
| | n=956 | Adjusted $R^2$=0.699 | | | RSE=0.409 | | | |
| log(s) | -0.985 | 0.370 | 0.203 | | -0.096 | | small | 1.723 |
| | (0.108) | (0.032) | (0.017) | | (0.010) | | | |
| | n=960 | Adjusted $R^2$=0.502 | | | RSE=0.420 | | | |
| **Time Series of Linear AR(1)** | | | | | | | | |
| L(P) | 0.377 | -0.103 | -0.269 | 0.019 | | 1.634 | small | |
| | (0.097) | (0.029) | (0.073) | (0.008) | | | | |
| | n=848 | Adjusted $R^2$=0.811 | | | RSE=0.605 | | | |
| log(s) | -1.453 | 0.547 | 0.282 | | -0.137 | | small | 1.703 |
| | (0.064) | (0.017) | (0.013) | | (0.007) | | | |
| | n=1024 | Adjusted $R^2$=0.725 | | | RSE=0.369 | | | |
| **Time Series of Linear MA(1)** | | | | | | | | |
| L(P) | 1.135 | -0.386 | -0.106 | | | 1.469 | small | |
| | (0.149) | (0.061) | (0.012) | | | | | |
| | n=953 | Adjusted $R^2$=0.745 | | | RSE=0.712 | | | |
| log(s) | -1.727 | 0.786 | 0.336 | | -0.197 | | small | 1.474 |
| | (0.097) | (0.035) | (0.015) | | (0.011) | | | |
| | n=1200 | Adjusted $R^2$=0.674 | | | RSE=0.502 | | | |
| **Time Series of Nonlinear AR(1)** | | | | | | | | |
| L(P) | -0.964 | 0.175 | | | -0.042 | small | small | |
| | (0.102) | (0.033) | | | (0.005) | | | |
| | n=1008 | Adjusted $R^2$=0.841 | | | RSE=0.439 | | | |
| log(s) | -1.246 | 0.470 | 0.192 | | -0.098 | | small | 1.636 |
| | (0.039) | (0.012) | (0.006) | | (0.004) | | | |
| | n=1040 | Adjusted $R^2$=0.828 | | | RSE=0.160 | | | |
| **Time Series of Nonlinear MA(1)** | | | | | | | | |
| L(P) | 0.750 | -0.255 | 0.319 | -0.034 | | 1.470 | 4.684 | |
| | (0.131) | (0.053) | (0.072) | (0.008) | | | | |
| | n=706 | Adjusted $R^2$=0.784 | | | RSE=0.545 | | | |
| log(s) | -1.562 | 0.731 | 0.329 | | -0.185 | | small | 1.447 |
| | (0.108) | (0.037) | (0.017) | | (0.012) | | | |
| | n=800 | Adjusted $R^2$=0.657 | | | RSE=0.464 | | | |

(continued)

**Table 5.2 (continued)**

| | L | $L^2$ | M | $M^2$ | LM | L* | M* | L*(M=3) |
|---|---|---|---|---|---|---|---|---|
| **Time Series of Threshold Autoregressive Process** | | | | | | | | |
| L(P) | -0.629 | 0.204 | 0.030 | | -0.056 | small | small | |
| | (0.076) | (0.023) | (0.011) | | (0.007) | | | |
| | n=2099 | Adjusted $R^2$=0.860 | | RSE=0.439 | | | | |
| log(s) | -1.136 | 0.459 | 0.208 | | -0.102 | | small | 1.570 |
| | (0.027) | (0.008) | (0.004) | | (0.003) | | | |
| | n=2240 | Adjusted $R^2$=0.824 | | RSE=0.158 | | | | |
| **Time Series of GARCH(1,1)** | | | | | | | | |
| L(P) | 0.195 | -0.046 | 0.058 | -0.014 | | 2.119 | 2.101 | |
| | (0.051) | (0.017) | (0.025) | (0.003) | | | | |
| | n=1633 | Adjusted $R^2$=0.925 | | RSE=0.282 | | | | |
| log(s) | -1.155 | 0.453 | 0.199 | | -0.095 | | small | 1.589 |
| | (0.033) | (0.010) | (0.005) | | (0.003) | | | |
| | n=1920 | Adjusted $R^2$=0.765 | | RSE=0.184 | | | | |

Note: The numbers in the columns of L, $L^2$, M, $M^2$, and LM are respectively the estimated coefficients of L, $L^2$, M, $M^2$, and LM from equations (5.4.1) and (5.4.2), other terms in the regression concerning sample size and parameter of the time series are omitted from the table;

The numbers in the bracket are the standard errors of the corresponding coefficients;

The numbers in the column of L* and M* are the optimum choices of L and M obtained from the estimated coefficients for the calculation of the BDS statistic;

The numbers in the column of L*(M=3) are the optimum choice of L obtained from the estimated coefficients for the calculation of the BDS statistic when M=3;

RSE denotes residual standard errors;

n denotes the sample size (number of experiments) from which the quoted regression was estimated.

## Table 5.3
## Regression Results of Response Surfaces of The BDS Statistic

**IID Time Series with Normal Distribution N(0,1)**

$$h\xi_1 = -\ 11.2/T\ \xi_1 \quad ,$$
(1.6)
[1.2]

n=12, adjusted $R^2$=0.4282, RSE=0.0419,

$$\log(s)\ \xi_2 = 22.6/T\ \xi_2 \quad ,$$
(1.9)
[1.3]

n=12, adjusted $R^2$=0.6171, RSE=0.0511,

$$(L(P) - L(0.025))\xi_3 = (202/T - 11581/T^2)\xi_3 \quad ,$$
(32) (3377)
[32] [3203]

n=12, adjusted $R^2$=0.7099, RSE=0.155.

**IID Time Series with Uniform Distribution U(0,1)**

$$h\xi_1 = (0.053 - 45.2/T + 2811/T^2)\xi_1 \quad ,$$
(0.021) (8.7) (706)
[0.014] [6.8] [584]

n=12, adjusted $R^2$=0.8497, RSE=0.0190,

$$\log(s)\ \xi_2 = (52.1/T - 1702/T^2)\xi_2 \quad ,$$
(7.1) (741)
[3.6] [485]

n=12, adjusted $R^2$=0.9128, RSE=0.0347,

$$(L(P) - L(0.025))\xi_3 = (0.528 + 69.7/T)\xi_3 \quad ,$$
(0.121) (14.3)
[0.107] [12.6]

n=12, adjusted $R^2$=0.6735, RSE=0.149.

**IID Time Series with Bimodal Mixture of Normal Distribution 0.5 N(0,1) + 0.5 N($\alpha,\beta^2$)**

$$h\xi_1 = (-0.0240\beta^2 - 1310/T^2 + 2.42\beta^2/T - 0.00000371\alpha^2T + 0.0000274\beta^2T)\xi_1 \quad ,$$
(0.0063) (162) (0.65) (0.00000156) (0.0000088)
[0.0040] [115] [0.39] [0.00000132] [0.0000061]

n=48, adjusted $R^2$=0.2347, RSE=0.0581,

$$\log(s)\ \xi_2 = (21.4/T + 0.0149\alpha^2 + 0.0418\beta^2 - 0.0525\alpha\beta + 2.33\alpha\beta/T - 3.49\beta^2/T$$
(1.6) (0.0013) (0.0088) (0.0079) (0.72) (0.86)
[1.3] [0.0016] [0.0145] [0.0105] [0.86] [1.32]

$$-\ 0.0000525\alpha T - 0.0000379\beta^2T + 0.0000417\alpha\beta T)\xi_2 \quad ,$$
(0.000020) (0.000013) (0.0000132)
[0.000015] [0.000015] [0.0000118]

n=48, adjusted $R^2$=0.8396, RSE=0.057,

$$(L(P) - L(0.025))\xi_3 = (-0.452\beta + 245/T + 0.0232\alpha^2 - 12454/T^2 - 2.39\alpha\beta/T$$
(0.100) (42) (0.0029) (3481) (0.85)
[0.079] [33] [0.0027] [2736] [0.46]

$$+\ 6.66\beta^2/T + 0.00069\beta T - 0.000088\alpha\beta T)\xi_3 \quad ,$$
(2.01) (0.00011) (0.000020)
[1.39] [0.000081] [0.000011]

n=48, adjusted $R^2$=0.7848, RSE=0.175.

(continued)

## Table 5.3 (continued)

**Linear AR(1) Process:** $x_t = \alpha\, x_{t-1} + \epsilon_t \;,\; \epsilon_t\; N(0,1)\;,$

$h\xi_1 = (-2.31\alpha + 28.3\alpha^2 - 1020\alpha^2/T + 0.0368\alpha^2 T)\,\xi_1\;,$

(0.25) (1.0) (88) (0.0013)
[0.21] [1.0] [87] [0.0013]

n=48, adjusted $R^2$=0.9977, RSE=0.136,

$\log(s)\;\xi_2 = (41.1/T + 2.00\alpha^2 - 1844/T^2 + 0.000418\alpha/T)\,\xi_2\;,$

(4.5) (0.053) (460) (0.000074)
[5.5] [0.039] [531] [0.000085]

n=48, adjusted $R^2$=0.9818, RSE=0.039,

$(L(P) - L(0.025))\,\xi_3 = (6.39\alpha + 93.1/T + 10.3\alpha^2 - 493\alpha/T + 0.0563\alpha^2 T)\,\xi_3\;,$

(0.57) (5.9) (0.94) (58) (0.0049)
[0.54] [4.9] [0.81] [50] [0.0043]

n=37, adjusted $R^2$=0.9949, RSE=0.118.

**Linear MA(1) Process:** $x_t = \alpha\, \epsilon_{t-1} + \epsilon_t \;,\; \epsilon_t - N(0,1)\;,$

$h\xi_1 = (-0.373 + 8.18\alpha - 1.27\alpha^2 - 289\alpha/T + 0.0106\alpha T)\,\xi_1\;,$

(0.11) (0.93) (0.66) (70) (0.00099)
[0.091] [0.83] [0.54] [65] [0.00104]

n=60, adjusted $R^2$=0.9745, RSE=0.489,

$\log(s)\;\xi_2 = (0.326\alpha + 35.5/T - 1595/T^2 + 34.1\alpha/T - 33.3\alpha^2/T + 0.000748\alpha T - 0.000683\alpha^2 T)\,\xi_2,$

(0.060) (4.8) (504) (8.3) (5.6) (0.000112) (0.000124)
[0.074] [7.9] [778] [7.8] [4.1] [0.000085] [0.000097]

n=60, adjusted $R^2$=0.9470, RSE=0.0327,

$(L(P) - L(0.025))\,\xi_3 = (0.422 + 8.50\alpha - 605\alpha^2/T + 0.0402\alpha^2 T)\,\xi_3\;,$

(0.091) (0.43) (42) (0.0033)
[0.120] [0.54] [47] [0.0028]

n=44, adjusted $R^2$=0.9799, RSE=0.243.

**Nonlinear AR(1) Process:** $x_t = \alpha\, x_{t-1}(1-x_{t-1})/(1+x_{t-1}^2) + \epsilon_t \;,\; \epsilon_t - N(0,1)\;,$

$h\xi_1 = (-0.447\alpha - 11.5/T + 3.56\alpha^2 - 109\alpha^2/T + 0.00481\alpha^2 T)\,\xi_1\;,$

(0.090) (1.9) (0.17) (14) (0.00018)
[0.079] [1.1] [0.17] [13] [0.00022]

n=52, adjusted $R^2$=0.9965, RSE=0.066,

$\log(s)\;\xi_2 = (31.7/T + 0.370\alpha^2 - 1084/T^2 + 0.000125\alpha T)\,\xi_2\;,$

(3.8) (0.020) (379) (0.000053)
[5.0] [0.017] [484] (0.000041)

n=52, adjusted $R^2$=0.9418, RSE=0.034,

$(L(P) - L(0.025))\,\xi_3 = (0.701\alpha + 175/T + 2.42\alpha^2 - 9513/T^2 - 93\alpha^2/T + 0.0097\alpha^2 T)\,\xi_3\;,$

(0.204) (24) (0.32) (2367) (29) (0.00053)
[0.159] [22] [0.28] [2173] [27] [0.00046]

n=50, adjusted $R^2$=0.9914, RSE=0.117.

**Nonlinear MA(1) Process:** $x_t = \alpha\, \epsilon_{t-1}\, \epsilon_{t-2} + \epsilon_t \;,\; \epsilon_t - N(0,1)\;,$

$h\xi_1 = (4.30\alpha - 14.7/T - 187\alpha/T + 0.00742\alpha T)\,\xi_1\;,$

(0.42) (7.2) (42) (0.00059)
[0.56] [5.1] [50] [0.00076]

n=40, adjusted $R^2$=0.9852, RSE=0.246,

$\log(s)\;\xi_2 = (0.716\alpha + 52.5/T - 0.472\alpha^2 - 3354/T^2)\,\xi_2\;,$

(0.067) (6.3) (0.074) (604)
[0.059] [8.5] [0.067] [800]

n=40, adjusted $R^2$=0.8666, RSE=0.046,

$(L(P) - L(0.025))\,\xi_3 = (11.7\alpha + 78.8/T - 10.1\alpha^2 - 616\alpha/T + 603\alpha^2/T + 0.0235\alpha^2 T)\,\xi_3\;,$

(0.69) (9.0) (1.13) (81) (106) (0.0035)
[0.51] [9.5] [0.93] [54] [90] [0.0017]

n=32, adjusted $R^2$=0.9875, RSE=0.154.

(continued)

## Table 5.3 (continued)

**Threshold Autoregressive Process:**

$$x_t = \alpha x_{t-1} + \epsilon_t \quad \text{if } x_{t-r} < 0 ,$$
$$= -\alpha x_{t-1} + \epsilon_t \quad \text{if } x_{t-r} > 0 , \quad \epsilon_t \sim N(0,1) ,$$

$$\begin{array}{lllllll} h\xi_1 = & (-4.31\alpha & + 9.90\alpha^2 & + 1.65\alpha r & - 84\alpha r/T & + 0.0165\alpha^2 T)\xi_1 , \\ & (0.47) & (0.91) & (0.16) & (18) & (0.0015) \\ & [0.44] & [0.99] & [0.18] & [22] & [0.0023] \end{array}$$

n=112, adjusted $R^2$=0.9409, RSE=0.37,

$$\begin{array}{lllllll} \log(s)\ \xi_2 = & (1.148\alpha^2 & + 0.00908r^2 & + 1927/T^2 & - 0.874r^2/T & + 0.000255\alpha T)\xi_2 , \\ & (0.059) & (0.0020) & (149) & (0.28) & (0.000088) \\ & [0.046] & [0.0019] & [91] & [0.24] & (0.000078) \end{array}$$

n=112, adjusted $R^2$=0.8505, RSE=0.065,

$$\begin{array}{llllll} (L(P) - L(0.025))\xi_3 = & (15.1\alpha^2 & + 29.2r/T & - 652\alpha^2/T & + 0.00327\alpha r T)\xi_3 , \\ & (0.88) & (2.9) & (97) & (0.00030) \\ & [1.16] & [3.1] & [131] & [0.00027] \end{array}$$

n=103, adjusted $R^2$=0.9133, RSE=0.412.

**GARCH(1,1) Process:**

$$x_t = h_t \epsilon_t , \quad \epsilon_t \sim N(0,1) ,$$
$$h^2_t = 1 + \alpha x^2_{t-1} + \beta h^2_{t-1} ,$$

$$\begin{array}{llllll} h\xi_1 = & (11.6\alpha & - 562\alpha/T & + 0.0129\alpha T & + 0.0275\alpha\beta T)\xi_1 , \\ & (0.33) & (33) & (0.0006) & (0.0016) \\ & [0.29] & [29] & [0.0005] & [0.0017] \end{array}$$

n=96, adjusted $R^2$=0.9964, RSE=0.122,

$$\begin{array}{lllllllll} \log(s)\ \xi_2 = & (54.3/T & - 3581/T^2 & + 2.22\alpha^2 & + 1.82\alpha\beta & + 137\alpha/T & - 337\alpha/T^2 & + 0.00069\alpha T)\xi_2 , \\ & (3.8) & (372) & (0.26) & (0.14) & (19) & (51) & (0.00011) \\ & [4.4] & [403] & [0.24] & [0.15] & [17] & [50] & [0.00011] \end{array}$$

n=96, adjusted $R^2$=0.9566, RSE=0.038,

$$\begin{array}{llllllll} (L(P)-L(0.025))\xi_3 = & (0.970 & + 13.31\alpha & - 8.22\alpha^2 & + 5.48\alpha\beta & - 432\alpha/T & + 0.0255\alpha T)\xi_3 , \\ & (0.049) & (0.69) & (1.13) & (0.48) & (43) & (0.0012) \\ & [0.040] & [0.66] & [0.99] & [0.49] & [41] & [0.0012] \end{array}$$

n=79, adjusted $R^2$=0.9903, RSE=0.116.

Note: h, s, P are respectively the mean, the standard deviation and rejection frequency of the BDS statistic;

$\xi_1 = N^{1/2}/s$ , $\xi_2 = (N/2)^{1/2}$ , $\xi_3 = (NP(1-P))^{1/2}$ are the heteroskedasticity transforms of the mean, the standard deviation and rejection frequency of BDS statistic;

N is the number of replication for each experiment;

(.), [.] respectively denote conventional and heteroskedasticity-consistent coefficient standard errors;

RSE denotes residual standard errors;

n denotes the sample size (number of experiments) from which the quoted regression was estimated.

Figure 5.1

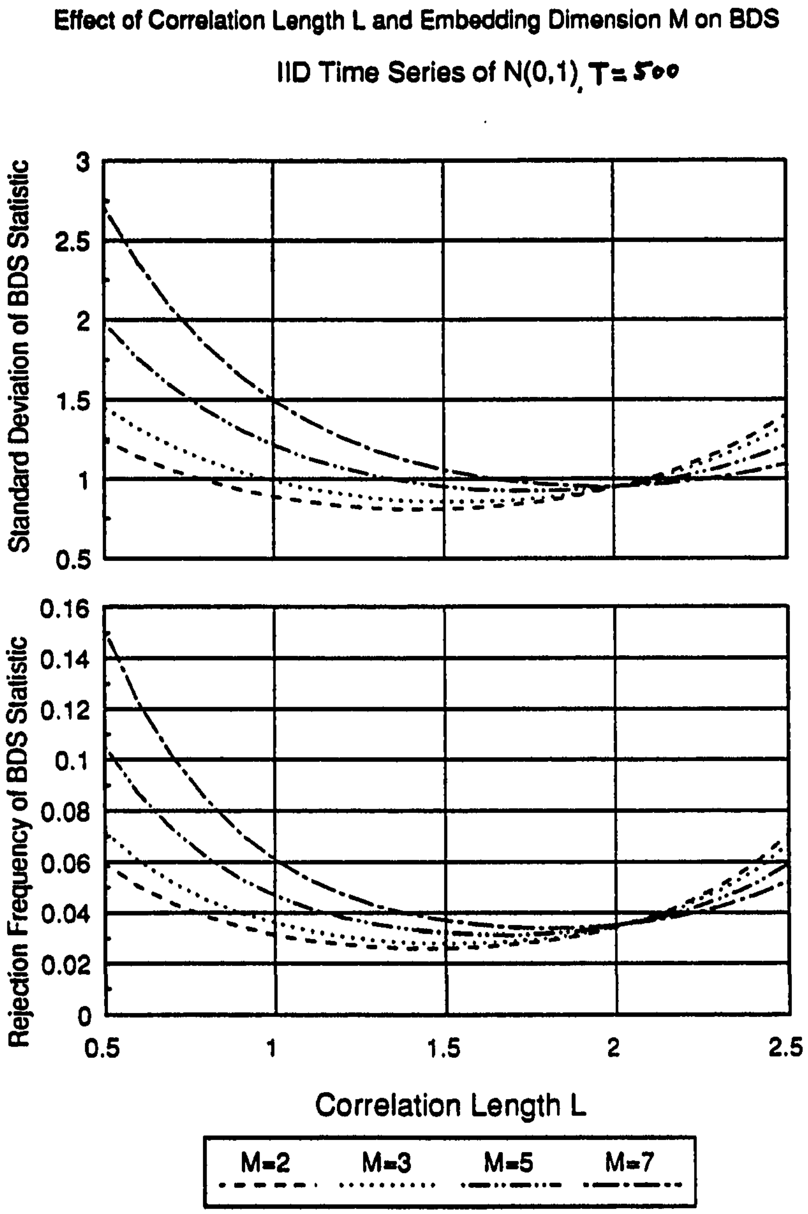

Effect of Correlation Length L and Embedding Dimension M on BDS
IID Time Series of N(0,1), T= 500
Standard Deviation of BDS Statistic
3
2.5
2
1.5
1
0.5
Rejection Frequency of BDS Statistic
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.5
1
1.5
2
2.5
Correlation Length L
M=2
M=3
M=5
M=7

Figure 5.2

Effect of Correlation Length L and Embedding Dimension M on BDS Time Series of Linear AR(1),α=0.3, T=500

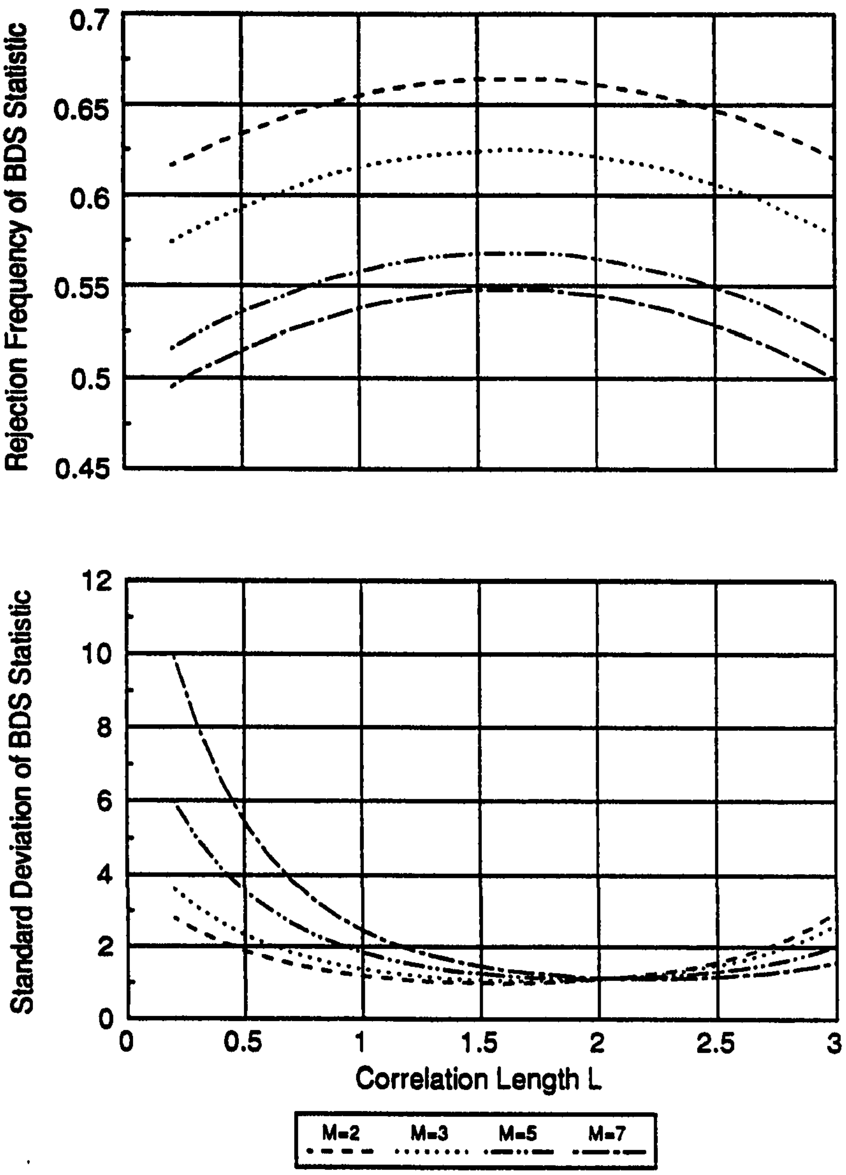


Figure 5.3

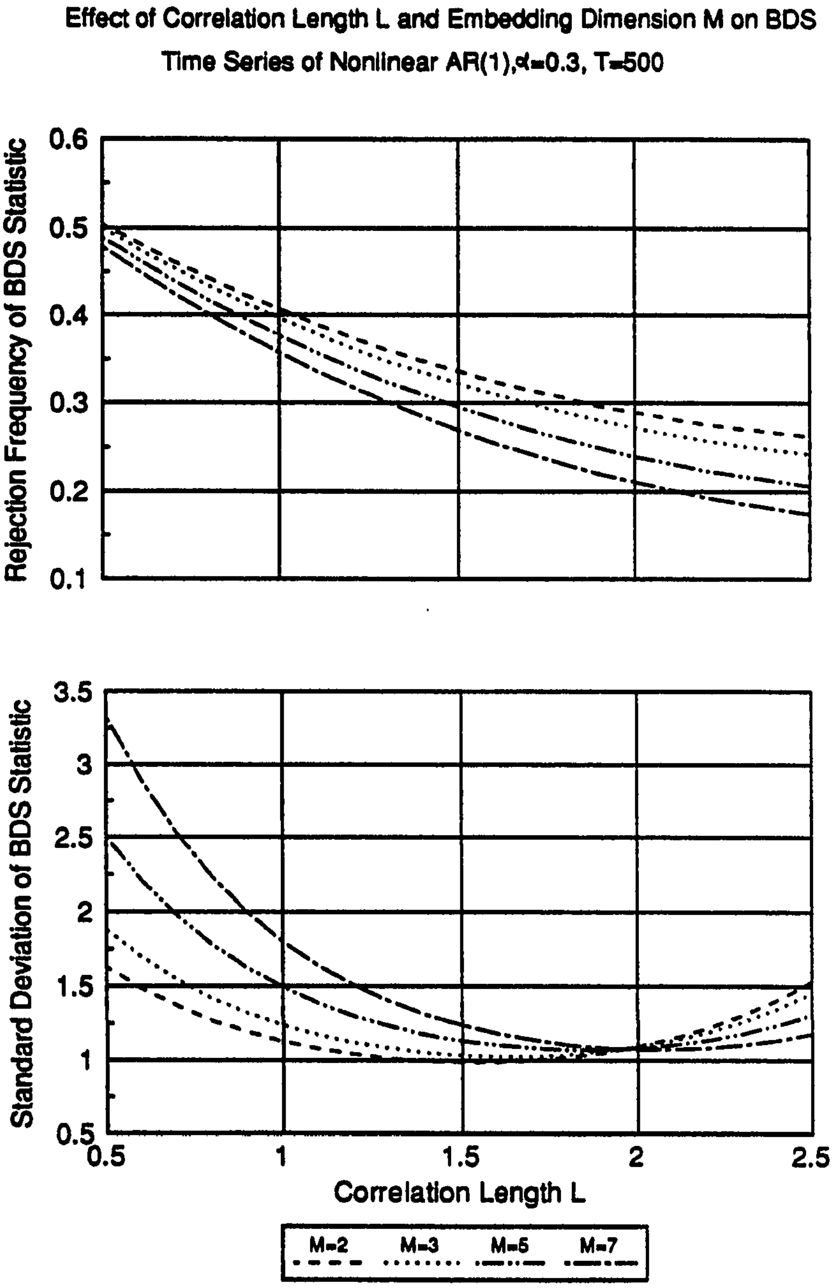

Effect of Correlation Length L and Embedding Dimension M on BDS
Time Series of Nonlinear AR(1),α=0.3, T=500
Rejection Frequency of BDS Statistic
0.6
0.5
0.4
0.3
0.2
0.1
Standard Deviation of BDS Statistic
3.5
3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
Correlation Length L
M=2
M=3
M=5
M=7

Figure 5.4

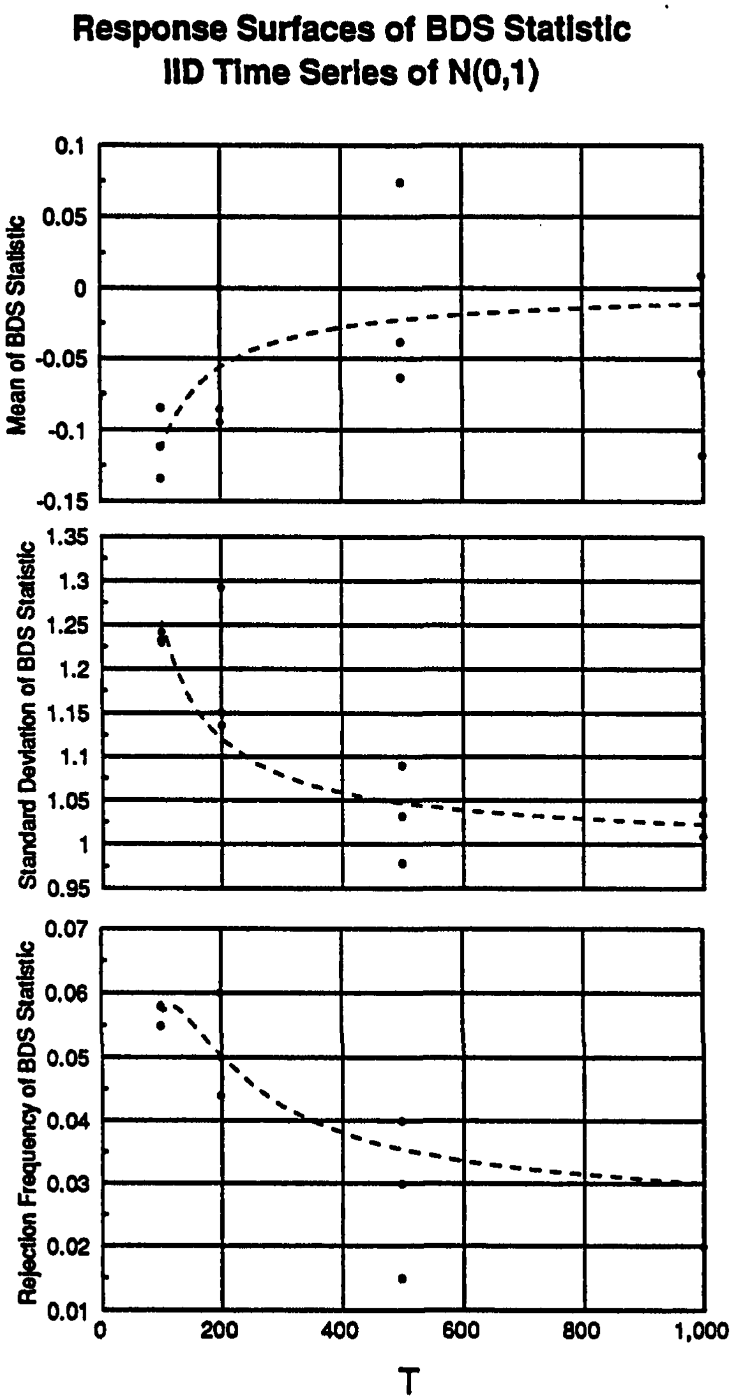

Response Surfaces of BDS Statistic
IID Time Series of N(0,1)
Mean of BDS Statistic
0.1
0.05
0
-0.05
-0.1
-0.15
Standard Deviation of BDS Statistic
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0.95
Rejection Frequency of BDS Statistic
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
200
400
600
800
1,000
T

Figure 5.5

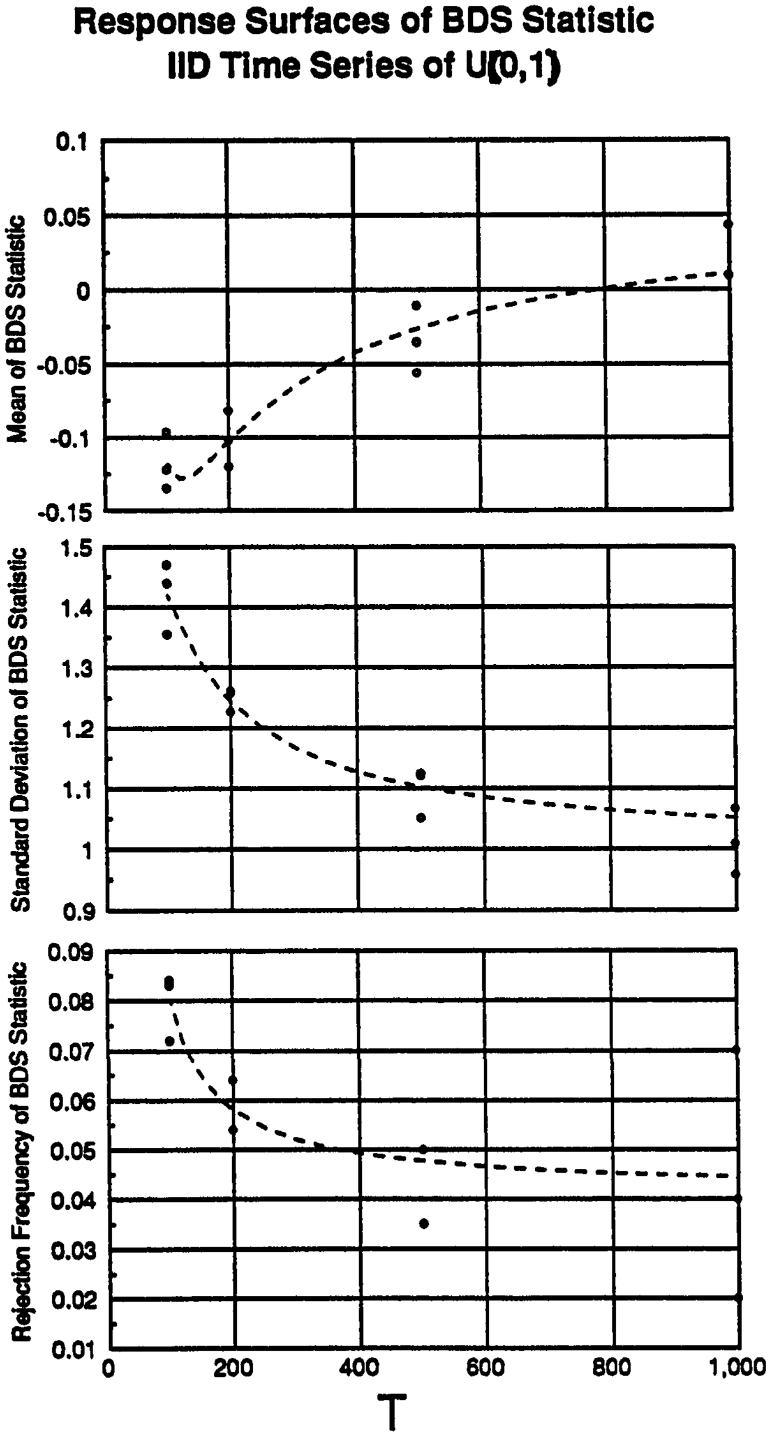

Response Surfaces of BDS Statistic
IID Time Series of U(0,1)
Mean of BDS Statistic
0.1
0.05
0
-0.05
-0.1
-0.15
Standard Deviation of BDS Statistic
1.5
1.4
1.3
1.2
1.1
1
0.9
Rejection Frequency of BDS Statistic
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
200
400
600
800
1,000
T

Figure 5.6

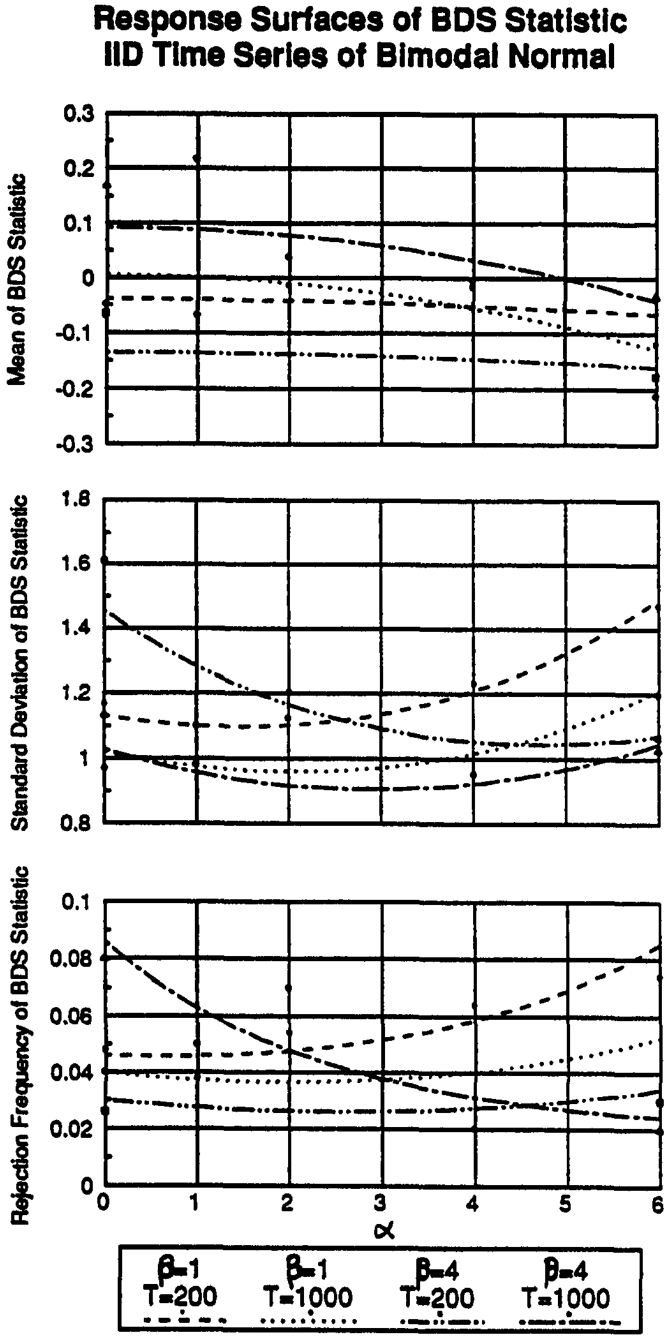

Response Surfaces of BDS Statistic
IID Time Series of Bimodal Normal
Mean of BDS Statistic
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
Standard Deviation of BDS Statistic
1.8
1.6
1.4
1.2
1
0.8
Rejection Frequency of BDS Statistic
0.1
0.08
0.06
0.04
0.02
0
0 1 2 3 4 5 6
α
β=1 T=200
β=1 T=1000
β=4 T=200
β=4 T=1000

Figure 5.7

**Response Surfaces of BDS Statistic**
**Time Series of Linear AR(1) Process**

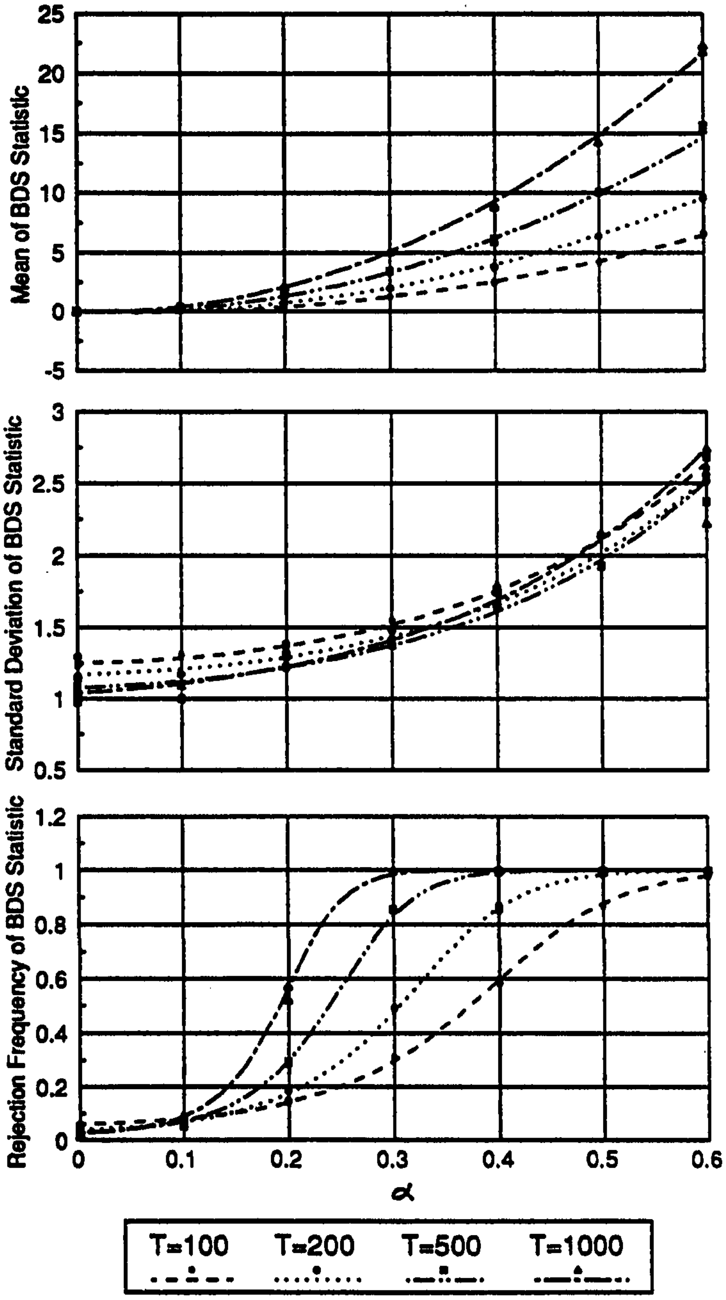


Figure 5.8

**Response Surfaces of BDS Statistic**
**Time Series of Linear MA(1) Process**

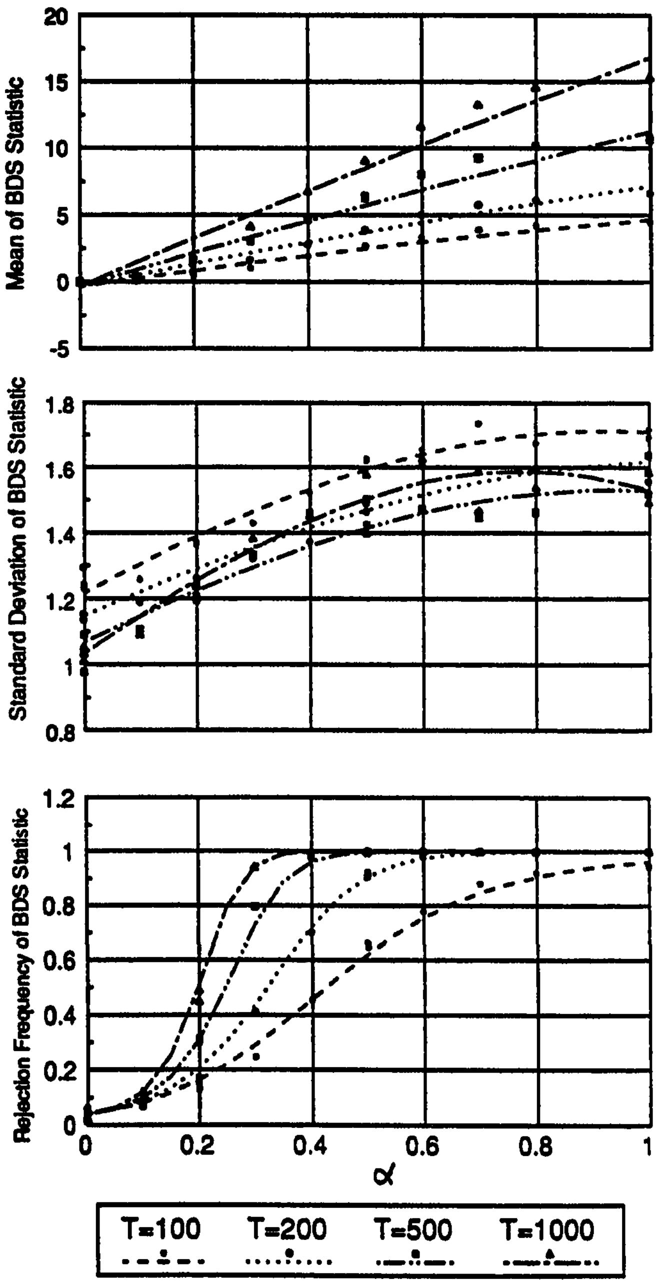


Figure 5.9

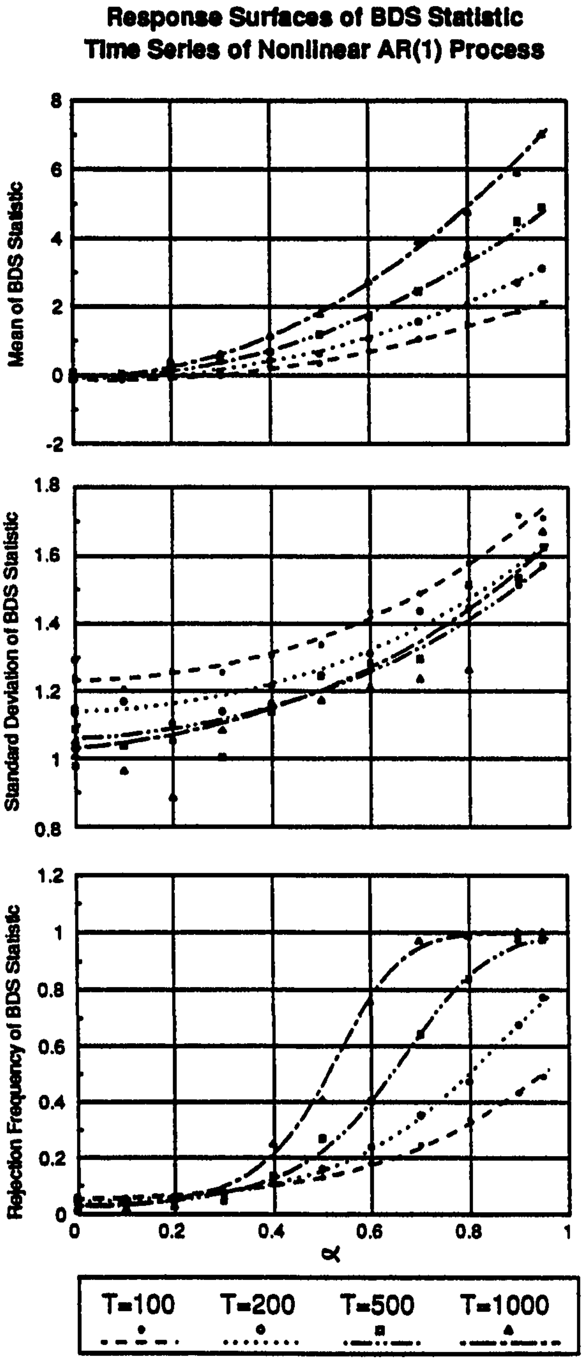
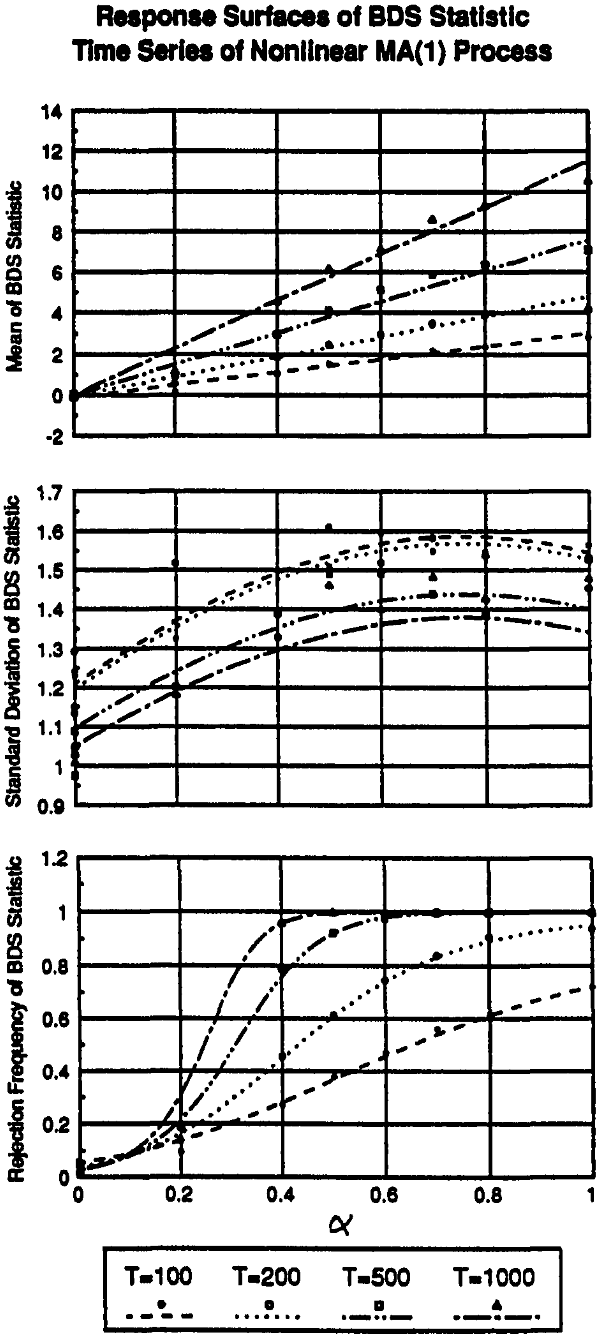

Response Surfaces of BDS Statistic
Time Series of Nonlinear AR(1) Process
Mean of BDS Statistic
8
6
4
2
0
-2
Standard Deviation of BDS Statistic
1.8
1.6
1.4
1.2
1
0.8
Rejection Frequency of BDS Statistic
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100
T=200
T=500
T=1000

Figure 5.10



Response Surfaces of BDS Statistic
Time Series of Nonlinear MA(1) Process
Mean of BDS Statistic
14
12
10
8
6
4
2
0
-2
Standard Deviation of BDS Statistic
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
Rejection Frequency of BDS Statistic
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100
T=200
T=500
T=1000

Figure 5.11

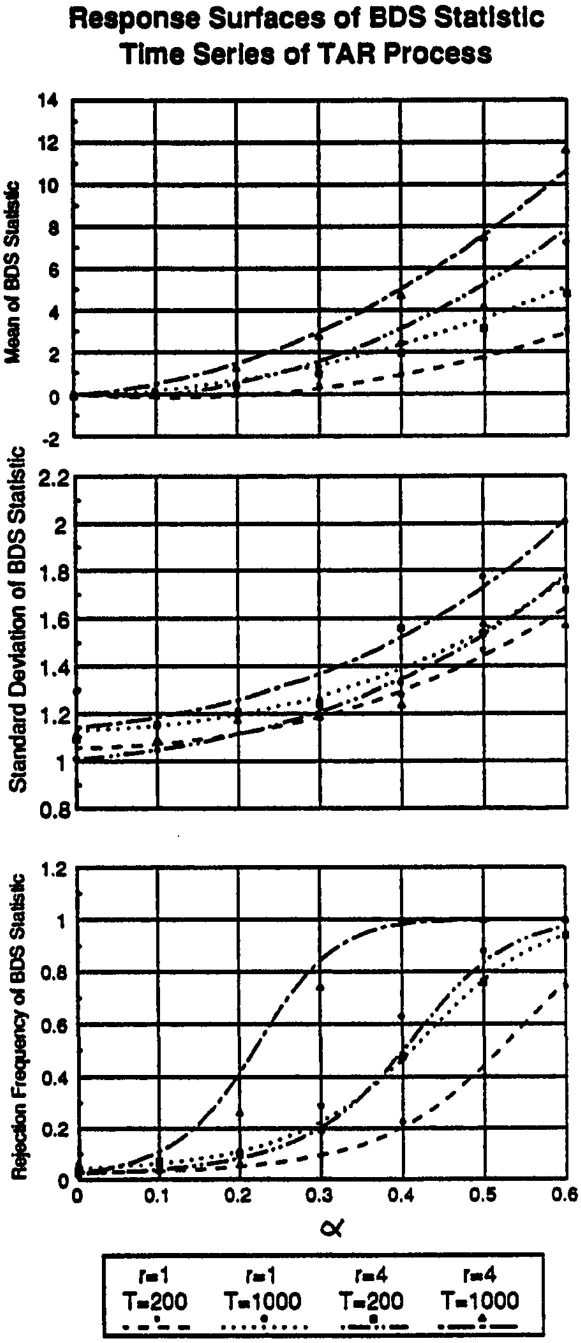
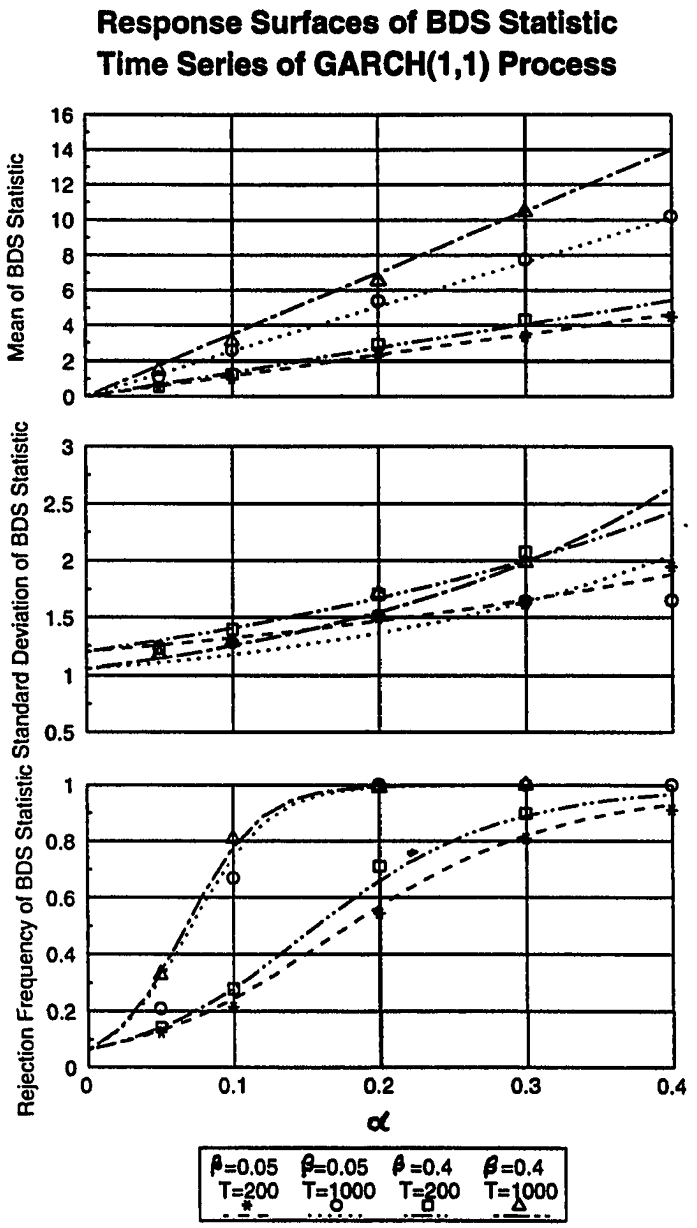

Response Surfaces of BDS Statistic
Time Series of TAR Process
Mean of BDS Statistic
Standard Deviation of BDS Statistic
Rejection Frequency of BDS Statistic
14
12
10
8
6
4
2
0
-2
2.2
2
1.8
1.6
1.4
1.2
1
0.8
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
α
r=1 T=200
r=1 T=1000
r=4 T=200
r=4 T=1000

Figure 5.12


Response Surfaces of BDS Statistic
Time Series of GARCH(1,1) Process
Mean of BDS Statistic
Standard Deviation of BDS Statistic
Rejection Frequency of BDS Statistic
0
2
4
6
8
10
12
14
16
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
α
β=0.05 T=200
β=0.05 T=1000
β=0.4 T=200
β=0.4 T=1000

## 5.5 Monte Carlo Study of The TAR-F Statistic

The TAR-F statistic of Tsay (1989) discussed in Section 4.5 is intended to give us a simple test statistic for detecting threshold autoregressive nonlinearity in time series and for identifying threshold lag of the threshold autoregressive model. The TAR-F statistic can be calculated from equations (4.5.6), (4.5.7), and (4.5.8).

Tsay (1989, 1991) studied the performance of the TAR-F statistic using Monte Carlo experiments. Tsay applied TAR-F test to DGPs of time series models such as linear AR model, threshold autoregressive (TAR) model, bilinear model, exponential AR model, concurrent nonlinear model (involving cross product of error term). Tsay found that, for linear time series, the TAR-F statistic has rejection frequency close to the that of F-distribution under null hypothesis. The TAR-F statistic has high rejection frequency against all nonlinear time series except the time series of exponential AR model. In these studies, more than one parameter values for each type of time series were used, but only two sample sizes were used in the first study and only one sample size was used in the second study. Furthermore, the TAR-F statistic was only calculated at threshold lag of one, so we do not know whether the TAR-F statistic can detect time series TAR process with threshold lag larger than one, and whether the TAR-F statistic can identify threshold lag of the TAR model.

The goal of this section then is two fold: a) to extend the scope of the Monte Carlo study of the TAR-F statistic to 9 types of time series process use 4 sample sizes and several parameter value of time series; and b) to use threshold lags of $d=1, 2, 3,$

4, and 5 in the TAR-F statistic. Therefore we investigate the performance of the TAR-F statistic on the TAR process and other nonlinear time series, and examine whether the TAR-F statistic can identify the threshold lag of the TAR model.

As discussed in the section 4.5, the parameters of the TAR-F statistic are the AR order p, the threshold lag d, and the sample size T. The other parameters which can be derived are b and h. The b is the number of the observation where the recursive regression starts, and h=max(1,p+1-d). In this section we select p=5, d=1, 2, 3, 4, 5. Following Tsay (1989), we use b=T/10+p+1=T/10+6. Therefore we have h=6-d, (T-b-d-p-h)=9T/10-17. At the sample sizes of 100, 200, 500, and 1000, the TAR-F statistics will have distributions of $F_{6,73}$, $F_{6,163}$, $F_{6,433}$, and $F_{6,883}$, respectively. The means and the standard deviations of these F-distributions can be calculated from equations (4.5.9) and (4.5.10), and they are reported in Table 5.4. At the rejection frequency of 5%, the percentile values of these F-distributions for the sample sizes of 100, 200, 500, and 1000 are respectively 2.225, 2.155, 2.120, and 2.105.

The DGPs of the time series processes and their parameter values used in Monte Carlo experiments are specified in Table 5.1. With these time series, the TAR-F statistics are calculated at the threshold lags of d=1, 2, 3, 4, and 5. Then the mean, the standard deviations, and the rejection frequencies of the replicated TAR-F statistics are obtained. For the types of time series which have smaller means and lower rejection frequencies of the TAR-F statistic, we only report the selected results in Table 5.4. But for other types of time series which have larger means and higher rejection frequencies of the TAR-F statistics, we present the response surfaces of the rejection frequencies in Table

5.5 and Figures 5.13 through 5.16, as well as the results of TAR-F statistic in Table 5.4.

For the IID time series with standard normal N(0,1) distribution, the means, the standard deviations, and the rejection frequencies of the TAR-F statistic are close to the means, the standard deviations, and the rejection frequencies of the F-distribution under the null hypothesis (see Table 5.4). The means are close to 1, the standard deviations are close to 0.6, and the rejection frequencies are close to 5%. The results shown in the table are calculated at threshold lag of d=1. The results at the threshold lag of d=2, 3, 4, and 5 are similar to the results at threshold lag of d=1, and thus are not presented in the table.

For the IID time series with uniform distribution, the means, the standard deviations, and the rejection frequencies of the TAR-F statistic are similar to their counter parts of the IID time series with normal distribution (see Table 5.4). The results at threshold lag of d=2, 3, 4, and 5 are also similar to the results at threshold lag of d=1 shown in the table.

The results of the IID time series with bimodal normal distribution are presented in Table 5.4 at threshold lag of d=1. The results at other threshold lags are also similar to those shown in the table and thus not presented in the table. The results at different means and standard deviations of the second mode of the normal distribution are similar and close to their corresponding values of the F-distribution under the null hypothesis. The only exception is at $\alpha=0$, $\beta=1$, and T=1000, where the standard deviation of the TAR-F statistic is about twice the standard deviation of the F-distribution under the null hypothesis. However, this is the results of the one single replication of the simulation which gives large TAR-F statistic. But the mean and the rejection frequency of the TAR-F statistic remain close to 1 and 5% respectively.

For the time series of linear AR(1) process, the results of TAR-F statistic at threshold lag of d=1 and AR coefficient $\alpha=0.8$ are shown in Table 5.4. The results at other threshold lags and other values of AR coefficient are similar to the results in the table and are not shown here. We can see that the means, the standard deviations, and the rejection frequencies of the TAR-F statistic are close to 1, 0.6 and 5% respectively. So the distribution of the TAR-F statistic is close to the F-distribution under the null hypothesis. For time series of linear MA(1) process, the results of TAR-F statistic are similar to those of linear AR(1) process (see Table 5.4).

For the time series of nonlinear AR(1) process, the TAR-F statistic can reject null hypothesis of linear process with large frequency (see Table 5.4 and Figure 5.13). At threshold lag of d=1, the mean, the standard deviation, and the rejection frequency of the TAR-F statistic increase when the AR coefficient increases and/or the sample size increases. For instance, when the AR coefficient $\alpha=0.8$ and the sample size T=1000, the rejection frequency of the TAR-F statistic is nearly 90%. However, at threshold lag of d=2, 3, 4, and 5, the TAR-F statistic has means close to 1, standard deviations close to 0.6, and rejection frequencies close to 5%. Thus the TAR-F statistic can rejected the null hypothesis of linear time series, and furthermore it can identify the nonlinearity in the form of lag one. This is what we wanted because the time series being tested are nonlinear, and the nonlinearity is in the form of lag one.

For the time series of nonlinear MA(1) process, the TAR-F statistic also can reject the null hypothesis of linear process with large frequency (see Table 5.4 and Figure 5.14). At threshold lag of d=1 and 2, the mean, the standard deviation, and the rejection frequency of the TAR-F statistic will increase as the MA coefficient increases and/or the sample size increases. When the MA coefficient $\alpha=0.8$, the TAR-F statistic can reject the null hypothesis 100% at the sample sizes of T=500 and T=1000. At threshold lag of d=3 and larger, the mean, the standard deviation, and the rejection frequency of the TAR-F statistic fall back to the values close to 1, 0.6, and 5% respectively. This phenomena can be explained by the nature of the nonlinear MA(1) process that the nonlinearity involves the first lag and the second lag of the time series, not the third lag or other lags beyond the third lag.

For the time series of threshold autoregressive process, the TAR-F statistic can reject the null hypothesis of linear process with large frequency as we expected (see Table 5.4 and Figure 5.15). The TAR-F statistic also can be used for identifying the threshold lag. For instance, when the threshold lag of the time series r=3 and the threshold AR coefficient of the time series $\alpha=0.6$, the mean of the TAR-F statistic will be very large at threshold lag of d=1 and 3. The large value of the mean of the TAR-F statistic at threshold lag of d=1 may be due to the threshold autoregressive process being in the form of first lag. And the large value of the mean of the TAR-F statistic at threshold lag d=3 is due to the threshold lag of the time series being at r=3. Although the mean of the TAR-F statistic at threshold lag of d=2, 4, 5 is also larger than the null hypothesis values of 1, it is much smaller than the mean at threshold lag of d=1 and 3. In this prospect, the TAR-F statistic also can identify the threshold lag of the threshold autoregressive process.

For the time series of threshold autoregressive process with threshold lag of r=3, we can see that the mean, the standard deviation, and the rejection frequency of the TAR-F statistic at threshold lag of d=3 will increase when the threshold AR coefficient increases and/or the sample size increases. For the time series of threshold autoregressive process with other threshold lags, the results of the TAR-F statistic are similar to that of the time series with threshold lag of r=3. For example, for time series of threshold autoregressive process with threshold lag of r=2, the TAR-F statistic has large mean at threshold lag of d=1 and 2 and has relatively smaller mean at threshold lag of d=3, 4, and 5. The mean, the standard deviation, and the rejection frequency of the TAR-F statistic at threshold lag of d=2 will also increase as the threshold AR coefficient increases and/or the sample size increases.

For the time series of GARCH(1,1) process, the TAR-F statistic can reject the null hypothesis of linear process (see Table 5.4 and Figure 5.16). The mean, the standard deviation, and the rejection frequency of the TAR-F statistic will increase if the first GARCH(1,1) coefficient $\alpha$ increases and/or the second GARCH(1,1) coefficient $\beta$ increases and/or the sample size T increases. However, the dependence of the mean, the standard deviation, and the rejection frequency of the TAR-F statistic on the second coefficient $\beta$ is not as strong as the dependence on the first coefficient $\alpha$. We also found that the TAR-F statistic has large mean, large standard deviation, and large rejection frequency at threshold lag of d=2, 3, 4, and 5 as well as at threshold lag of d=1 compared to the mean, the standard deviation, and rejection frequency of F-distribution under the null hypothesis. But as the threshold lag increases, the mean, the standard deviation, and the rejection frequency of the TAR-F statistic will all decrease.

In summary, we extended Monte Carlo study of the TAR-F statistic to nine types of time series, using AR order of p=5, and using threshold lag of d=1, 2, 3, 4, and 5 instead of just d=1 in the TAR-F statistic. The results show that the means, the standard deviations, and the rejection frequencies of the TAR-F statistic for IID time series, for time series of linear AR(1) process, and for time series of linear MA(1) process studied are close to the mean, the standard deviation, and the rejection frequency of the F-distribution under the null hypothesis of linear process. For the time series of nonlinear AR(1) process, the time series of nonlinear MA(1) process, the time series of threshold autoregressive process, and the time series of GARCH(1,1) process the TAR-F statistic gives us large mean and large rejection frequency at certain threshold lags. The results also shows that the TAR-F statistic can be used for identifying the threshold lag of the threshold autoregressive model. For these nonlinear time series, the mean, the standard deviation, and the rejection frequency of the TAR-F statistic will increase as the parameters of the time series increase and/or the sample size increases.

Table 5.4
Results of Monte Carlo Experiments for TAR-F Statistic

| T | Mean | Mean* | STD | STD* | P | P* |
|---|---|---|---|---|---|---|
| IID Time Series with Normal Distribution | | | | | | |
| | | | d=1 | | | |
| 100 | 1.006 | 1.023 | 0.594 | 0.617 | 0.046 | 0.050 |
| 200 | 0.983 | 1.011 | 0.582 | 0.597 | 0.044 | 0.050 |
| 500 | 0.980 | 1.004 | 0.599 | 0.585 | 0.045 | 0.050 |
| 1000 | 1.022 | 1.002 | 0.641 | 0.581 | 0.060 | 0.050 |
| IID Time Series with Uniform Distribution | | | | | | |
| | | | d=1 | | | |
| 100 | 1.022 | 1.023 | 0.615 | 0.617 | 0.042 | 0.050 |
| 200 | 0.996 | 1.011 | 0.543 | 0.597 | 0.024 | 0.050 |
| 500 | 0.997 | 1.004 | 0.583 | 0.585 | 0.045 | 0.050 |
| 1000 | 0.962 | 1.002 | 0.531 | 0.581 | 0.050 | 0.050 |
| IID Time Series with Bimodal Distribution | | | | | | |
| | | | d=1 | | | |
| $\alpha$=0, $\beta$=1 | | | | | | |
| 100 | 1.002 | 1.023 | 0.623 | 0.617 | 0.042 | 0.050 |
| 200 | 0.997 | 1.011 | 0.635 | 0.597 | 0.058 | 0.050 |
| 500 | 1.019 | 1.004 | 0.562 | 0.585 | 0.040 | 0.050 |
| 1000 | 1.128 | 1.002 | 1.170 | 0.581 | 0.060 | 0.050 |
| $\alpha$=0, $\beta$=4 | | | | | | |
| 100 | 1.032 | 1.023 | 0.688 | 0.617 | 0.067 | 0.050 |
| 200 | 0.989 | 1.011 | 0.608 | 0.597 | 0.050 | 0.050 |
| 500 | 1.031 | 1.004 | 0.601 | 0.585 | 0.070 | 0.050 |
| 1000 | 0.934 | 1.002 | 0.709 | 0.581 | 0.060 | 0.050 |
| $\alpha$=6, $\beta$=1 | | | | | | |
| 100 | 0.988 | 1.023 | 0.550 | 0.617 | 0.037 | 0.050 |
| 200 | 0.990 | 1.011 | 0.548 | 0.597 | 0.034 | 0.050 |
| 500 | 0.957 | 1.004 | 0.573 | 0.585 | 0.045 | 0.050 |
| 1000 | 0.968 | 1.002 | 0.492 | 0.581 | 0.010 | 0.050 |
| a=$\alpha$, $\beta$=4 | | | | | | |
| 100 | 0.977 | 1.023 | 0.619 | 0.617 | 0.047 | 0.050 |
| 200 | 0.963 | 1.011 | 0.590 | 0.597 | 0.038 | 0.050 |
| 500 | 1.035 | 1.004 | 0.580 | 0.585 | 0.045 | 0.050 |
| 1000 | 1.041 | 1.002 | 0.545 | 0.581 | 0.040 | 0.050 |
| Time Series of Linear AR(1) Process | | | | | | |
| | | d=1 | $\alpha$=0.0 | | | |
| 100 | 0.948 | 1.023 | 0.590 | 0.617 | 0.041 | 0.050 |
| 200 | 0.963 | 1.011 | 0.594 | 0.597 | 0.032 | 0.050 |
| 500 | 0.995 | 1.004 | 0.571 | 0.585 | 0.040 | 0.050 |
| 1000 | 1.094 | 1.002 | 0.662 | 0.581 | 0.100 | 0.050 |
| Time Series of Linear MA(1) Process | | | | | | |
| | | d=1 | $\alpha$=0.8 | | | |
| 100 | 0.990 | 1.023 | 0.628 | 0.617 | 0.050 | 0.050 |
| 200 | 0.959 | 1.011 | 0.580 | 0.597 | 0.038 | 0.050 |
| 500 | 0.996 | 1.004 | 0.611 | 0.585 | 0.055 | 0.050 |
| 1000 | 0.979 | 1.002 | 0.602 | 0.581 | 0.040 | 0.050 |

(continued)

Table 5.4 (continued)

| T | Mean | Mean* | STD | STD* | P | P* |
|---|---|---|---|---|---|---|
| Time Series of Nonlinear AR(1) Process $\alpha$=0.8 | | | | | | |
| d=1 | | | | | | |
| 100 | 1.358 | 1.023 | 0.796 | 0.617 | 0.131 | 0.050 |
| 200 | 1.618 | 1.011 | 0.876 | 0.597 | 0.252 | 0.050 |
| 500 | 2.350 | 1.004 | 1.202 | 0.585 | 0.490 | 0.050 |
| 1000 | 3.716 | 1.002 | 1.540 | 0.581 | 0.860 | 0.050 |
| d=2 | | | | | | |
| 100 | 0.996 | 1.023 | 0.591 | 0.617 | 0.041 | 0.050 |
| 200 | 1.008 | 1.011 | 0.625 | 0.597 | 0.058 | 0.050 |
| 500 | 1.047 | 1.004 | 0.640 | 0.585 | 0.050 | 0.050 |
| 1000 | 0.966 | 1.002 | 0.603 | 0.581 | 0.060 | 0.050 |
| Time Series of Nonlinear MA(1) Process $\alpha$=0.8 | | | | | | |
| d=1 | | | | | | |
| 100 | 2.766 | 1.023 | 1.564 | 0.617 | 0.570 | 0.050 |
| 200 | 4.378 | 1.011 | 2.038 | 0.597 | 0.864 | 0.050 |
| 500 | 9.305 | 1.004 | 2.898 | 0.585 | 1.000 | 0.050 |
| 1000 | 16.974 | 1.002 | 4.494 | 0.581 | 1.000 | 0.050 |
| d=2 | | | | | | |
| 100 | 2.914 | 1.023 | 1.623 | 0.617 | 0.619 | 0.050 |
| 200 | 4.777 | 1.011 | 2.121 | 0.597 | 0.920 | 0.050 |
| 500 | 10.942 | 1.004 | 3.464 | 0.585 | 1.000 | 0.050 |
| 1000 | 20.638 | 1.002 | 5.215 | 0.581 | 1.000 | 0.050 |
| d=3 | | | | | | |
| 100 | 1.119 | 1.023 | 0.747 | 0.617 | 0.081 | 0.050 |
| 200 | 1.168 | 1.011 | 0.742 | 0.597 | 0.086 | 0.050 |
| 500 | 1.128 | 1.004 | 0.671 | 0.585 | 0.060 | 0.050 |
| 1000 | 1.243 | 1.002 | 0.733 | 0.581 | 0.140 | 0.050 |
| Time Series of GARCH(1,1) Process d=1 | | | | | | |
| $\alpha$=0.3, $\beta$=0.05 | | | | | | |
| 100 | 1.502 | 1.023 | 0.989 | 0.617 | 0.173 | 0.050 |
| 200 | 1.697 | 1.011 | 1.081 | 0.597 | 0.266 | 0.050 |
| 500 | 1.771 | 1.004 | 1.026 | 0.585 | 0.305 | 0.050 |
| 1000 | 2.060 | 1.002 | 1.351 | 0.581 | 0.370 | 0.050 |
| $\alpha$=0.05, $\beta$=0.3 | | | | | | |
| 100 | 1.042 | 1.023 | 0.623 | 0.617 | 0.046 | 0.050 |
| 200 | 1.088 | 1.011 | 0.648 | 0.597 | 0.062 | 0.050 |
| 500 | 1.132 | 1.004 | 0.660 | 0.585 | 0.090 | 0.050 |
| 1000 | 1.146 | 1.002 | 0.686 | 0.581 | 0.090 | 0.050 |
| $\alpha$=0.3, $\beta$=0.3 | | | | | | |
| 100 | 1.637 | 1.023 | 1.095 | 0.617 | 0.233 | 0.050 |
| 200 | 1.798 | 1.011 | 1.196 | 0.597 | 0.290 | 0.050 |
| 500 | 1.938 | 1.004 | 1.184 | 0.585 | 0.375 | 0.050 |
| 1000 | 2.300 | 1.002 | 1.768 | 0.581 | 0.350 | 0.050 |

(continued)

Table 5.4 (continued)

| T | Mean | Mean* | STD | STD* | P | P* |
|---|---|---|---|---|---|---|
| Time Series of Threshold AR Process | | | | | | |
| | | | r=3, α=0.6 | | | |
| d=1 | | | | | | |
| 100 | 4.623 | 1.023 | 2.101 | 0.617 | 0.902 | 0.050 |
| 200 | 8.564 | 1.011 | 2.941 | 0.597 | 1.000 | 0.050 |
| 500 | 21.347 | 1.004 | 4.794 | 0.585 | 1.000 | 0.050 |
| 1000 | 42.623 | 1.002 | 7.387 | 0.581 | 1.000 | 0.050 |
| d=2 | | | | | | |
| 100 | 1.220 | 1.023 | 0.766 | 0.617 | 0.089 | 0.050 |
| 200 | 1.331 | 1.011 | 0.785 | 0.597 | 0.140 | 0.050 |
| 500 | 1.381 | 1.004 | 0.795 | 0.585 | 0.165 | 0.050 |
| 1000 | 1.532 | 1.002 | 0.850 | 0.581 | 0.200 | 0.050 |
| d=3 | | | | | | |
| 100 | 4.423 | 1.023 | 2.074 | 0.617 | 0.887 | 0.050 |
| 200 | 8.322 | 1.011 | 3.022 | 0.597 | 0.998 | 0.050 |
| 500 | 20.260 | 1.004 | 4.729 | 0.585 | 1.000 | 0.050 |
| 1000 | 40.767 | 1.002 | 6.663 | 0.581 | 1.000 | 0.050 |
| d=4 | | | | | | |
| 100 | 1.292 | 1.023 | 0.826 | 0.617 | 0.123 | 0.050 |
| 200 | 1.515 | 1.011 | 0.991 | 0.597 | 0.200 | 0.050 |
| 500 | 2.273 | 1.004 | 1.362 | 0.585 | 0.430 | 0.050 |
| 1000 | 3.253 | 1.002 | 1.443 | 0.581 | 0.770 | 0.050 |
| d=5 | | | | | | |
| 100 | 1.310 | 1.023 | 0.840 | 0.617 | 0.126 | 0.050 |
| 200 | 1.549 | 1.011 | 0.988 | 0.597 | 0.228 | 0.050 |
| 500 | 2.297 | 1.004 | 1.332 | 0.585 | 0.455 | 0.050 |
| 1000 | 3.217 | 1.002 | 1.476 | 0.581 | 0.740 | 0.050 |

Note: T is the sample size of the TAR-F statistic;

Mean, STD, P are respectively the mean, the standard deviation, and the rejection frequency of the TAR-F statistic;

Mean*, STD*, P* are respectively the mean, the standard deviation, and the rejection frequency of the F-distribution under the null hypothesis;

d is the threshold lag of the Tsay F-statistic;

α, β, and r are the parameters of the time series.

## Table 5.5
## Regression Results of The Response Surface
## The Rejection Frequency of The TAR-F Statistic

**Time Series of Nonlinear AR(1) Process**

d=1

$$(L(P) - L(0.05))\xi_3 = (1.99\,\alpha - 20.4/T - 96\,\alpha/T + 0.00833\,\alpha T - 0.00511\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $1/T$ | $\alpha/T$ | $\alpha T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (0.21) | (7.3) | (23) | (0.00054) | (0.00065) |
| (0.15) | [5.6] | [18] | [0.00065] | [0.00083] |

n=52, Adjusted $R^2$=0.9736, RSE=0.185

**Time Series of Nonlinear MA(1) Process**

d=1

$$(L(P) - L(0.05))\xi_3 = (19.97\,\alpha - 55.0\,\alpha^2 - 1145\,\alpha/T + 3607\,\alpha^2/T + 0.135\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $\alpha^2$ | $\alpha/T$ | $\alpha^2/T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (0.74) | (5.2) | (82) | (346) | (0.018) |
| [0.62] | [3.7] | [67] | [237] | [0.013] |

n=38, Adjusted $R^2$=0.9839, RSE=0.170

d=2

$$(L(P) - L(0.05))\xi_3 = (15.8\,\alpha - 35.3/T - 15.4\,\alpha^2 - 791\,\alpha/T + 924\,\alpha^2/T + 0.0189\,\alpha T)\xi_3\,,$$

| $\alpha$ | $1/T$ | $\alpha^2$ | $\alpha/T$ | $\alpha^2/T$ | $\alpha T$ |
|---|---|---|---|---|---|
| (1.6) | (12.0) | (1.1) | (145) | (128) | (0.0035) |
| [1.7] | [10.4] | [1.3] | [159] | [142] | [0.0038] |

n=39, Adjusted $R^2$=0.9769, RSE=0.211

**Time Series of Threshold Autoregressive Process**

r=1, d=1

$$(L(P) - L(0.05))\xi_3 = (9.9\,\alpha - 17.6\,\alpha^2 - 608\,\alpha/T + 1274\,\alpha^2/T + 0.0683\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $\alpha^2$ | $\alpha/T$ | $\alpha^2/T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (1.3) | (2..6) | (156) | (278) | (0.0085) |
| [1.1] | [1.8] | [137] | [234] | [0.0048] |

n=23, Adjusted $R^2$=0.9826, RSE=0.173

r=2, d=2

$$(L(P) - L(0.05))\xi_3 = (10.6\,\alpha - 25.0\,\alpha^2 - 787\,\alpha/T + 1977\,\alpha^2/T + 0.136\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $\alpha^2$ | $\alpha/T$ | $\alpha^2/T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (1.1) | (3.4) | (124) | (291) | (0.012) |
| [1.0] | [2.7] | [114] | [216] | [0.012] |

n=21, Adjusted $R^2$=0.9948, RSE=0.109

r=3, d=3

$$(L(P) - L(0.05))\xi_3 = (6.89\,\alpha - 28.4/T - 305\,\alpha/T + 0.0101\,\alpha T + 0.0666\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $1/T$ | $\alpha/T$ | $\alpha T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (1.15) | (12.4) | (80) | (0.0024) | (0.0093) |
| [0.90] | [15.0] | [63] | [0.0020] | [0.0088] |

n=21, Adjusted $R^2$=0.9913, RSE=0.144

**Time Series of GARCH(1,1) Process**

d=1

$$(L(P) - L(0.05))\xi_3 = (9.29\,\alpha - 16.6/T - 4.94\,\alpha^2 - 264\,\alpha/T + 324\,\alpha\beta/T + 0.00500\,\alpha\beta T)\xi_3\,,$$

| $\alpha$ | $1/T$ | $\alpha^2$ | $\alpha/T$ | $\alpha\beta/T$ | $\alpha\beta T$ |
|---|---|---|---|---|---|
| (0.35) | (5.8) | (0.88) | (28) | (54) | (0.00100) |
| [0.28] | [6.2] | [0.82] | [28] | [40] | [0.00097] |

n=96, Adjusted $R^2$=0.9734, RSE=0.123

d=2

$$(L(P) - L(0.05))\xi_3 = (3.48\,\alpha - 33.7/T + 10.89\,\alpha\beta - 566\,\alpha\beta/T + 0.00382\,\alpha^2 T)\xi_3\,,$$

| $\alpha$ | $1/T$ | $\alpha\beta$ | $\alpha\beta/T$ | $\alpha^2 T$ |
|---|---|---|---|---|
| (0.18) | (4.9) | (0.80) | (102) | (0.00087) |
| [0.15] | [4.5] | [0.76] | [114] | [0.00061] |

n=96, Adjusted $R^2$=0.9559, RSE=0.157

Note: L(P)=log(P(1-P)), where P is the rejection frequency of the TAR-F statistic;
$\xi_3 = (NP(1-P))^{1/2}$ is the heteroskedasticity transforms of the rejection frequency of the TAR-F statistic, where N is the number of replication for each experiment;
(.), [.] respectively denote conventional and heteroskedasticity-consistent coefficient standard errors;
RSE denotes residual standard errors;
n denotes the sample size (number of experiments) from which the quoted regression was estimated;
d is the threshold lag of the Tsay F-statistic;
$\alpha$, $\beta$, and r are the parameters of the time series.

Figure 5.13

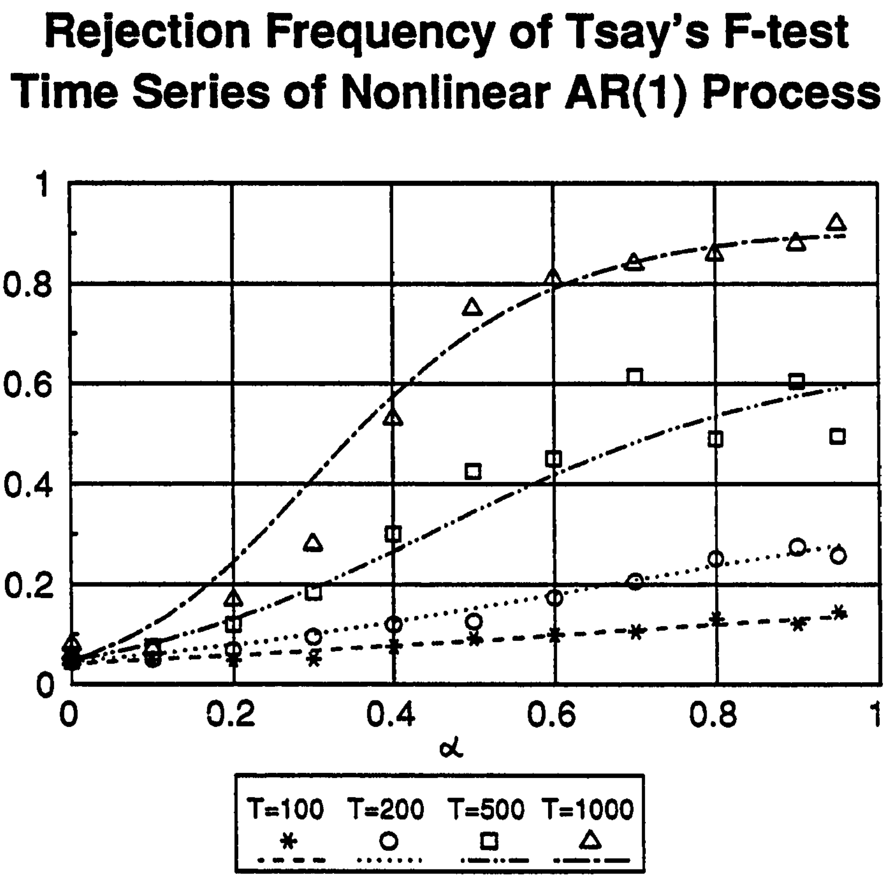

Rejection Frequency of Tsay's F-test
Time Series of Nonlinear AR(1) Process
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100 T=200 T=500 T=1000

Figure 5.14

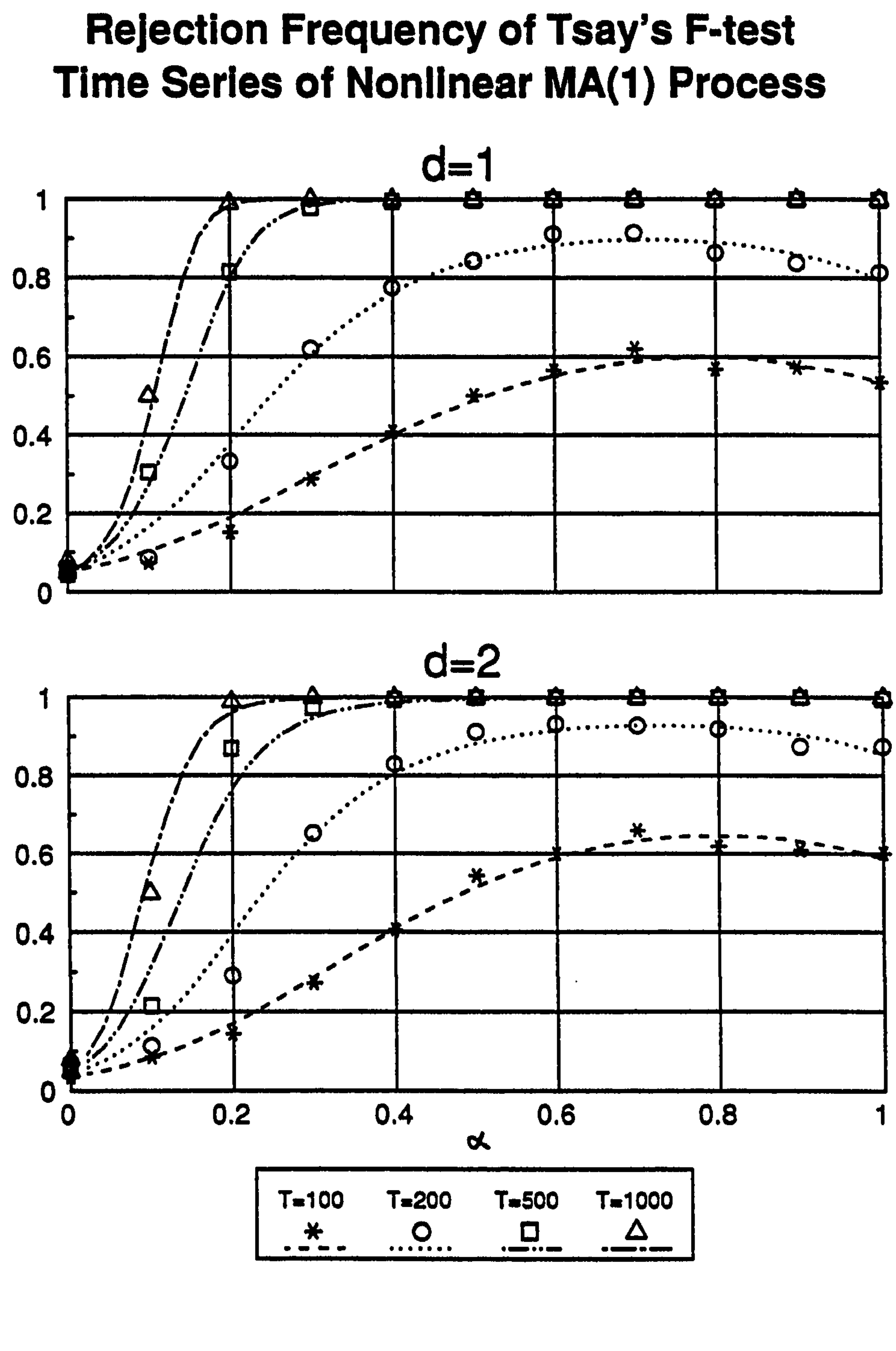

Rejection Frequency of Tsay's F-test
Time Series of Nonlinear MA(1) Process
d=1
d=2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
α
T=100
T=200
T=500
T=1000

Figure 5.15

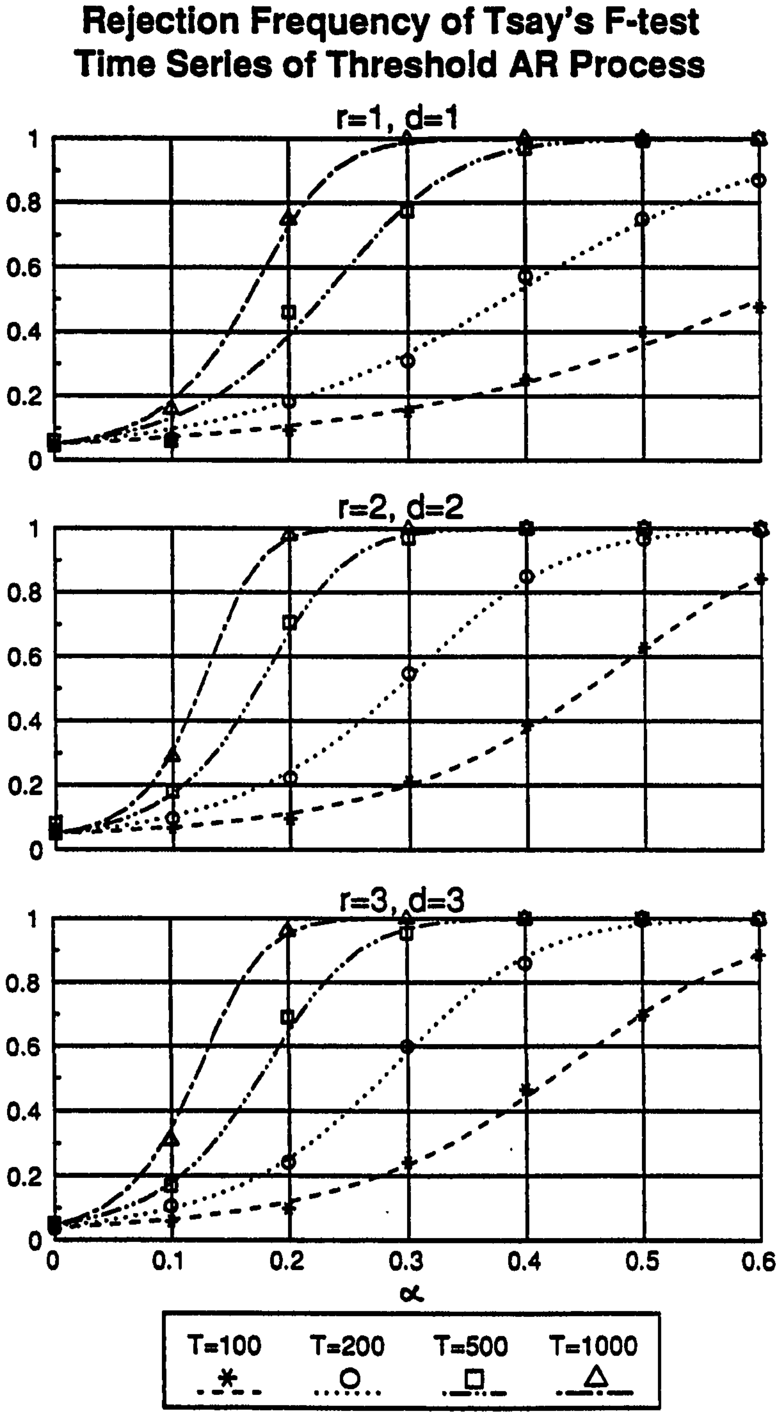
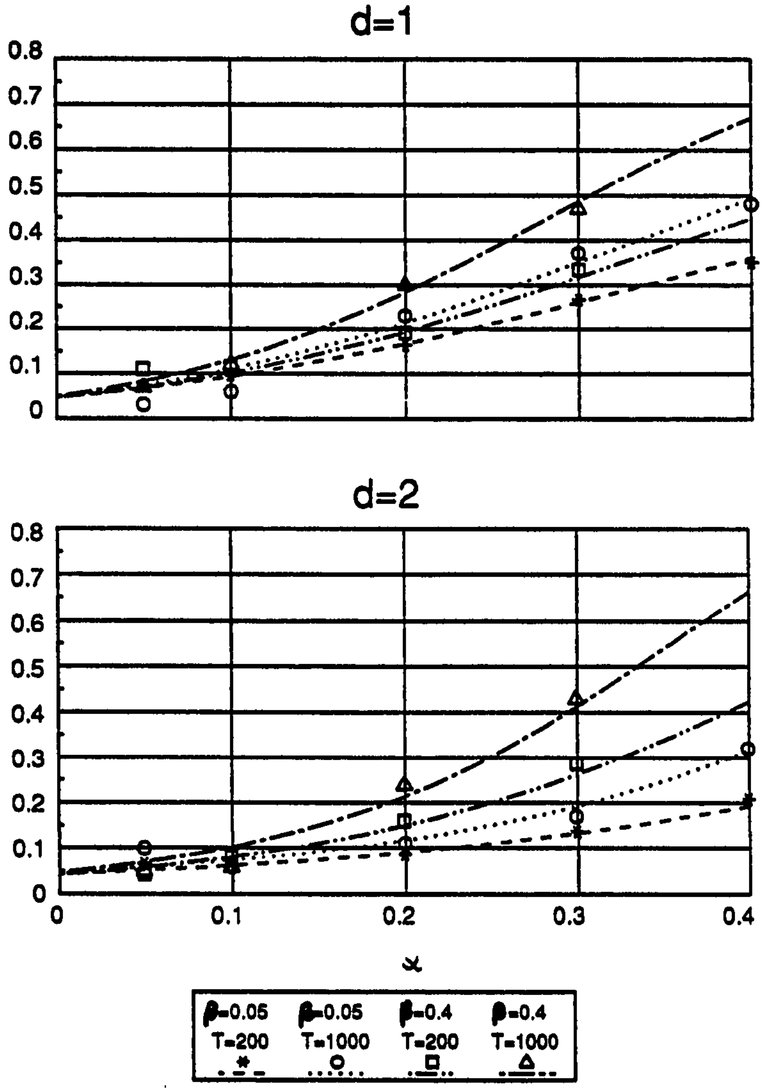

Rejection Frequency of Tsay's F-test
Time Series of Threshold AR Process
r=1, d=1
r=2, d=2
r=3, d=3
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
α
T=100
T=200
T=500
T=1000

Figure 5.16

**Rejection Frequency of Tsay's F-test**
**Time Series of GARCH(1,1) Process**



## 5.6 Monte Carlo Study of The $Q^2$ Statistic

The $Q^2$ statistic studied here is the portmanteau test statistic proposed by McLeod and Li (1983), which was discussed in Section 4.6. The $Q^2$ statistic is intended to be used for identifying the nonlinear time series, such as the bilinear time series and ARCH type nonlinear time series, where the Ljung-Box autocorrelation method fails. But McLeod and Li (1983) only did small sample simulation of the $Q^2$ statistic for the time series of linear AR(1) process. In this section we extend the Monte Carlo study of the $Q^2$ statistic to several types of time series at four sample sizes. We also examine whether the selection of the lag of the $Q^2$ statistic has any effect on the rejection frequency.

The lags of the $Q^2$ statistic used in our study are $p=5$ and $p=10$. So under the null hypothesis of the time series has no serial dependence in its squared value, the $Q^2$ statistic will have distributions of $\chi^2(5)$ and $\chi^2(10)$ at the lag of $p=5$ and $p=10$ respectively. The means of the $\chi^2$ distributions then will be 5 and 10, the standard deviations of the $\chi^2$ distributions will be 3.16 and 4.47 for $\chi^2(5)$ and $\chi^2(10)$ respectively. At the rejection frequency of 5%, the percentile of these $\chi^2$ distributions will be 11.07 and 18.31 for $\chi^2(5)$ and $\chi^2(10)$ respectively. So the $Q^2$ statistic of the simulated time series will be compared to these values under the null hypothesis.

The types of time series processes and the parameters of the time series studied in this section are specified in Table 5.1. The $Q^2$ statistics are calculated for simulated samples of these time series at the lag of $p=5$ and 10. Then the mean, the standard deviation, and the rejection frequency of the replicated $Q^2$ statistics are obtained.

Although the simulations are performed at the lag of p=5 and p=10, we found the results of the rejection frequency of the $Q^2$ statistic at lag of p=10 are very similar to the results at lag of p=5, so we only report the results at lag of p=5 of the $Q^2$ statistic here. For the IID time series which have smaller means and lower rejection frequencies of the $Q^2$ statistic, we report the selected results in Table 5.6. But for other types of time series which have larger means and higher rejection frequencies of the $Q^2$ statistics, we also present the response surfaces of the rejection frequencies in Table 5.7 and Figures 5.17 through 5.22, as well as the results in Table 5.6.

For the IID time series with normal distribution N(0,1), the mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic are close to the mean, the standard deviation, and the rejection frequency of the $\chi^2$ distribution under the null hypothesis, which are 5, 3.16, and 5% respectively (see Table 5.6). With the four sample sizes considered, we found the increase of the sample size has small effect on the $Q^2$ statistic. The results of the $Q^2$ statistic for the IID time series with uniform distribution U(0,1) are very similar to the results for the IID time series with normal distribution (see Table 6.6).

For the IID time series with bimodal normal distribution, although the results are obtained at different values of mean and different values of standard deviation of the second mode of the normal distribution, the results are all similar (see Table 5.6). The results at different sample sizes are also similar. The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic are close to 5, 3.16, and 5% respectively, which are consistent with the null hypothesis.

For the time series of linear AR(1) process, the $Q^2$ statistic can reject the null hypothesis with large frequency (see Tables 5.6, 5.7 and Figure 5.17). The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic are larger than the values of 5, 3.16, and 5% under the null hypothesis. The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic will increase when the AR coefficient increases and/or the sample size increases.

For the time series of linear MA(1) process, the time series of nonlinear AR(1) process, and the time series of nonlinear MA(1) process, the properties of the $Q^2$ statistic have similar behavior as for the time series of the linear AR(1) process (see Tables 5.6, 5.7, Figures 5.18, 5.19, 5.20). The only difference is that the magnitudes of the mean, of the standard deviation, and of the rejection frequency of the $Q^2$ statistic are different for each type of time series. For these time series, the $Q^2$ statistics can reject the null hypothesis with large frequency. The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic will increase when the coefficient of the time series increases and/or the sample size increases.

For the time series of threshold autoregressive process, the $Q^2$ statistic can reject the null hypothesis with large frequency (see Tables 5.6, 5.7, and Figure 5.21). The results are nearly same at both threshold lag of $r=1$ and $r=3$ of the threshold model. And these results are similar to the results for time series of linear AR(1) process. That is, the mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic will increase when the coefficient of the time series increases and/or the sample size increases.

For the time series of GARCH(1,1) process, the $Q^2$ statistic has the ability to reject the null hypothesis with large frequency (see Tables 5.6, 5.7, and Figure 5.22). The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic will increase when the first coefficient of the GARCH(1,1) process increases and/or the second coefficient of the GARCH(1,1) process increases and/or the sample size increases. The influence of the first coefficient of the GARCH(1,1) process is stronger than the influence of the second coefficient of the GARCH(1,1) process on the mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic.

In summary, the $Q^2$ statistic has nearly same rejection frequency at the lag of $p=5$ and $p=10$ for the time series studied. We conclude that as long as the lag of the $Q^2$ statistic is larger than the lag involved in the time series process, the choice of the lag of the $Q^2$ statistic will only have small effect on the rejection frequency. The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic for the IID time series are close to their values of the $\chi^2$ distribution under the null hypothesis. For the time series with serial dependence, the $Q^2$ statistic can reject the null hypothesis with large frequency. The mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic will increase when the coefficients of the time series increase and/or the sample size increases.

Table 5.6
Results of Monte Carlo Experiments for $Q^2$ Statistic

| T | Mean | Mean* | STD | STD* | P | P* |
|---|---|---|---|---|---|---|
| IID Time Series with Normal Distribution | | | | | | |
| 100 | 4.750 | 5.000 | 3.120 | 3.162 | 0.040 | 0.050 |
| 200 | 4.790 | 5.000 | 3.080 | 3.162 | 0.034 | 0.050 |
| 500 | 4.640 | 5.000 | 2.840 | 3.162 | 0.030 | 0.050 |
| 1000 | 5.070 | 5.000 | 3.240 | 3.162 | 0.060 | 0.050 |
| IID Time Series with Uniform Distribution | | | | | | |
| 100 | 4.990 | 5.000 | 3.210 | 3.162 | 0.040 | 0.050 |
| 200 | 5.400 | 5.000 | 3.450 | 3.162 | 0.074 | 0.050 |
| 500 | 5.380 | 5.000 | 3.650 | 3.162 | 0.080 | 0.050 |
| 1000 | 5.140 | 5.000 | 3.360 | 3.162 | 0.050 | 0.050 |
| IID Time Series with Bimodal Distribution | | | | | | |
| a=0, b=1 | | | | | | |
| 100 | 4.590 | 5.000 | 3.030 | 3.162 | 0.032 | 0.050 |
| 200 | 4.800 | 5.000 | 3.020 | 3.162 | 0.044 | 0.050 |
| 500 | 4.610 | 5.000 | 2.770 | 3.162 | 0.035 | 0.050 |
| 1000 | 4.990 | 5.000 | 4.220 | 3.162 | 0.060 | 0.050 |
| a=0, b=4 | | | | | | |
| 100 | 4.330 | 5.000 | 3.360 | 3.162 | 0.049 | 0.050 |
| 200 | 4.600 | 5.000 | 3.040 | 3.162 | 0.046 | 0.050 |
| 500 | 4.900 | 5.000 | 3.840 | 3.162 | 0.070 | 0.050 |
| 1000 | 4.980 | 5.000 | 3.030 | 3.162 | 0.030 | 0.050 |
| a=6, b=1 | | | | | | |
| 100 | 5.120 | 5.000 | 3.330 | 3.162 | 0.053 | 0.050 |
| 200 | 4.890 | 5.000 | 2.970 | 3.162 | 0.050 | 0.050 |
| 500 | 5.300 | 5.000 | 3.410 | 3.162 | 0.065 | 0.050 |
| 1000 | 4.670 | 5.000 | 2.600 | 3.162 | 0.010 | 0.050 |
| a=6, b=4 | | | | | | |
| 100 | 4.790 | 5.000 | 3.030 | 3.162 | 0.035 | 0.050 |
| 200 | 4.760 | 5.000 | 3.250 | 3.162 | 0.050 | 0.050 |
| 500 | 4.990 | 5.000 | 3.170 | 3.162 | 0.040 | 0.050 |
| 1000 | 5.470 | 5.000 | 3.430 | 3.162 | 0.070 | 0.050 |
| Time Series of Linear AR(1) Process | | | | | | |
| a=0.6 | | | | | | |
| 100 | 17.190 | 5.000 | 13.110 | 3.162 | 0.614 | 0.050 |
| 200 | 32.730 | 5.000 | 21.430 | 3.162 | 0.902 | 0.050 |
| 500 | 76.110 | 5.000 | 35.110 | 3.162 | 1.000 | 0.050 |
| 1000 | 148.870 | 5.000 | 53.830 | 3.162 | 1.000 | 0.050 |
| Time Series of Linear MA(1) Process | | | | | | |
| a=0.6 | | | | | | |
| 100 | 8.010 | 5.000 | 5.590 | 3.162 | 0.210 | 0.050 |
| 200 | 12.020 | 5.000 | 7.420 | 3.162 | 0.464 | 0.050 |
| 500 | 23.260 | 5.000 | 11.070 | 3.162 | 0.905 | 0.050 |
| 1000 | 43.370 | 5.000 | 20.010 | 3.162 | 1.000 | 0.050 |

(continued)

Table 5.6 (continued)

| T | Mean | Mean* | STD | STD* | P | P* |
|---|---|---|---|---|---|---|
| Time Series of Nonlinear AR(1) Process | | | | | | |
| a=0.8 | | | | | | |
| 100 | 6.510 | 5.000 | 5.280 | 3.162 | 0.128 | 0.050 |
| 200 | 8.090 | 5.000 | 6.080 | 3.162 | 0.236 | 0.050 |
| 500 | 13.320 | 5.000 | 7.140 | 3.162 | 0.570 | 0.050 |
| 1000 | 19.890 | 5.000 | 10.870 | 3.162 | 0.770 | 0.050 |
| Time Series of Nonlinear MA(1) Process | | | | | | |
| a=0.6 | | | | | | |
| 100 | 11.090 | 5.000 | 8.690 | 3.162 | 0.377 | 0.050 |
| 200 | 17.820 | 5.000 | 12.850 | 3.162 | 0.628 | 0.050 |
| 500 | 39.560 | 5.000 | 21.410 | 3.162 | 0.965 | 0.050 |
| 1000 | 76.180 | 5.000 | 30.890 | 3.162 | 1.000 | 0.050 |
| Time Series of Threshold Autoregressive Process | | | | | | |
| r=1, a=0.6 | | | | | | |
| 100 | 17.580 | 5.000 | 13.990 | 3.162 | 0.635 | 0.050 |
| 200 | 32.870 | 5.000 | 19.750 | 3.162 | 0.936 | 0.050 |
| 500 | 71.070 | 5.000 | 28.890 | 3.162 | 1.000 | 0.050 |
| 1000 | 156.130 | 5.000 | 52.250 | 3.162 | 1.000 | 0.050 |
| r=3, a=0.6 | | | | | | |
| 100 | 17.400 | 5.000 | 12.700 | 3.162 | 0.621 | 0.050 |
| 200 | 31.670 | 5.000 | 20.340 | 3.162 | 0.896 | 0.050 |
| 500 | 79.520 | 5.000 | 32.710 | 3.162 | 1.000 | 0.050 |
| 1000 | 151.300 | 5.000 | 48.150 | 3.162 | 1.000 | 0.050 |
| Time Series of GARCH(1,1) Process | | | | | | |
| a=0.3, b=0.05 | | | | | | |
| 100 | 11.150 | 5.000 | 10.560 | 3.162 | 0.342 | 0.050 |
| 200 | 21.590 | 5.000 | 19.610 | 3.162 | 0.678 | 0.050 |
| 500 | 55.980 | 5.000 | 50.730 | 3.162 | 0.970 | 0.050 |
| 1000 | 102.750 | 5.000 | 87.390 | 3.162 | 1.000 | 0.050 |
| a=0.05, b=0.3 | | | | | | |
| 100 | 5.040 | 5.000 | 3.560 | 3.162 | 0.052 | 0.050 |
| 200 | 5.720 | 5.000 | 4.720 | 3.162 | 0.088 | 0.050 |
| 500 | 7.090 | 5.000 | 5.690 | 3.162 | 0.185 | 0.050 |
| 1000 | 8.310 | 5.000 | 5.850 | 3.162 | 0.260 | 0.050 |
| a=0.3, b=0.3 | | | | | | |
| 100 | 14.340 | 5.000 | 13.390 | 3.162 | 0.471 | 0.050 |
| 200 | 29.110 | 5.000 | 29.370 | 3.162 | 0.746 | 0.050 |
| 500 | 71.130 | 5.000 | 53.990 | 3.162 | 0.990 | 0.050 |
| 1000 | 176.800 | 5.000 | 153.530 | 3.162 | 1.000 | 0.050 |

Note: T is the sample size of the $Q^2$ statistic;

Mean, STD, P are respectively the mean, the standard deviation, and the rejection frequency of the $Q^2$ statistic;

Mean*, STD*, P* are respectively the mean, the standard deviation, and the rejection frequency of the $\chi^2$-distribution under the null hypothesis;

$\alpha$, $\beta$, and r are the parameters of the time series.

**Table 5.7**
**Regression Results of The Response Surfaces**
**Rejection Frequency of $Q^2$ Statistic**

**Time Series of Linear AR(1) Process**

$$(L(P) - L(0.05))\xi_3 = (-24.7/T + 11.80\,\alpha^2 - 375\,\alpha^2/T + 0.0241\,\alpha^2 T)\xi_3\,,$$

| | $-24.7/T$ | $11.80\,\alpha^2$ | $-375\,\alpha^2/T$ | $0.0241\,\alpha^2T$ |
|---|---|---|---|---|
| | (5.9) | (1.06) | (90) | (0.0023) |
| | [5.4] | [0.34] | [42] | [0.0009] |

n=36, Adjusted $R^2$=0.9905, RSE=0.167

**Time Series of Linear MA(1) Process**

$$(L(P) - L(0.05))\xi_3 = (6.81\,\alpha - 50/T - 5.25\,\alpha^2 - 248\,\alpha/T + 259\,\alpha^2/T + 0.0137\,\alpha^2 T)\xi_3\,,$$

| | $6.81\,\alpha$ | $-50/T$ | $-5.25\,\alpha^2$ | $-248\,\alpha/T$ | $259\,\alpha^2/T$ | $0.0137\,\alpha^2T$ |
|---|---|---|---|---|---|---|
| | (0.71) | (14) | (0.86) | (99) | (107) | (0.0019) |
| | [0.71] | [13] | [0.88] | [91] | [102] | [0.0015] |

n=47, Adjusted $R^2$=0.9607, RSE=0.252

**Time Series of Nonlinear AR(1) Process**

$$(L(P) - L(0.05))\xi_3 = (-15.0/T + 2.01\,\alpha^2 - 64\,\alpha/T + 0.00516\,\alpha^2 T)\xi_3\,,$$

| | $-15.0/T$ | $2.01\,\alpha^2$ | $-64\,\alpha/T$ | $0.00516\,\alpha^2T$ |
|---|---|---|---|---|
| | (7.0) | (0.19) | (20) | (0.00033) |
| | [5.1] | [0.16] | [16] | [0.00033] |

n=51, Adjusted $R^2$=0.9787, RSE=0.171

**Time Series of Nonlinear MA(1) Process**

$$(L(P) - L(0.05))\xi_3 = (9.39\,\alpha - 50/T - 8.9\,\alpha^2 - 262\,\alpha/T + 372\,\alpha^2/T + 0.0184\,\alpha^2 T)\xi_3\,,$$

| | $9.39\,\alpha$ | $-50/T$ | $-8.9\,\alpha^2$ | $-262\,\alpha/T$ | $372\,\alpha^2/T$ | $0.0184\,\alpha^2T$ |
|---|---|---|---|---|---|---|
| | (0.71) | (14) | (1.0) | (95) | (108) | (0.0031) |
| | [0.79] | [12] | [1.1] | [91] | [96] | [0.0038] |

n=44, Adjusted $R^2$=0.9663, RSE=0.229

**Time Series of Threshold Autoregressive Process**

r=1

$$(L(P) - L(0.05))\xi_3 = (2.09\,\alpha - 28.2/T + 8.3\,\alpha^2 - 227\,\alpha/T - 0.0040\,\alpha T + 0.0336\,\alpha^2 T)\xi_3\,,$$

| | $2.09\,\alpha$ | $-28.2/T$ | $8.3\,\alpha^2$ | $-227\,\alpha/T$ | $-0.0040\,\alpha T$ | $0.0336\,\alpha^2T$ |
|---|---|---|---|---|---|---|
| | (0.66) | (13.2) | (1.5) | (60) | (0.0020) | (0.0065) |
| | [0.50] | [7.5] | [1.4] | [56] | [0.0016] | [0.0053] |

n=25, Adjusted $R^2$=0.9902, RSE=0.144

r=3

$$(L(P) - L(0.05))\xi_3 = (0.254 + 6.51\,\alpha^2 - 4291/T^2 - 0.0057\,\alpha T + 0.0463\,\alpha^2 T)\xi_3\,,$$

| | $0.254$ | $6.51\,\alpha^2$ | $-4291/T^2$ | $-0.0057\,\alpha T$ | $0.0463\,\alpha^2T$ |
|---|---|---|---|---|---|
| | (0.097) | (0.52) | (1074) | (0.0018) | (0.0054) |
| | [0.080] | [0.43] | [930] | [0.0011] | [0.0041] |

n=25, Adjusted $R^2$=0.9898, RSE=0.145

**Time Series of GARCH(1,1) Process**

$$(L(P) - L(0.05))\xi_3 = (18.6\,\alpha - 38.2/T - 31.9\,\alpha^2 - 800\,\alpha/T + 1786\,\alpha^2/T + 324\,\alpha\beta/T + 0.0126\,\alpha T + 0.041\,\alpha^2 T + 0.0207\,\alpha\beta T\,,$$

| | $18.6\,\alpha$ | $-38.2/T$ | $-31.9\,\alpha^2$ | $-800\,\alpha/T$ | $1786\,\alpha^2/T$ | $324\,\alpha\beta/T$ |
|---|---|---|---|---|---|---|
| | (1.14) | (6.6) | (4.8) | (120) | (399) | (65) |
| | [0.99] | [5.9] | [3.8] | [107] | [337] | [61] |

| | $0.0126\,\alpha T$ | $0.041\,\alpha^2T$ | $0.0207\,\alpha\beta T$ |
|---|---|---|---|
| | (0.0023) | (0.013) | (0.0034) |
| | [0.0022] | [0.010] | [0.0031] |

n=83, Adjusted $R^2$=0.9907, RSE=0.112

Note: L(P)=log(P/(1-P)), where P is the rejection frequency of the $Q^2$ statistics;
$\xi_3 = (NP(1-P))^{1/2}$ is the heteroskedasticity transforms of the rejection frequency of the $Q^2$ statistics, where N is the number of replication for each experiment;
(.), [.] respectively denote conventional and heteroskedasticity-consistent coefficient standard errors;
RSE denotes residual standard errors;
n denotes the sample size (number of experiments) from which the quoted regression was estimated.

Figure 5.17

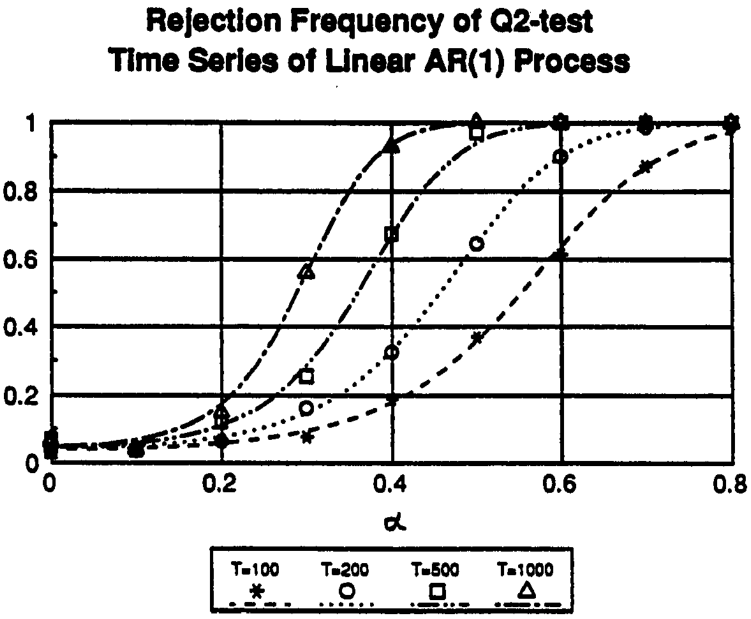

Rejection Frequency of Q2-test
Time Series of Linear AR(1) Process
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
α
T=100
T=200
T=500
T=1000

Figure 5.18

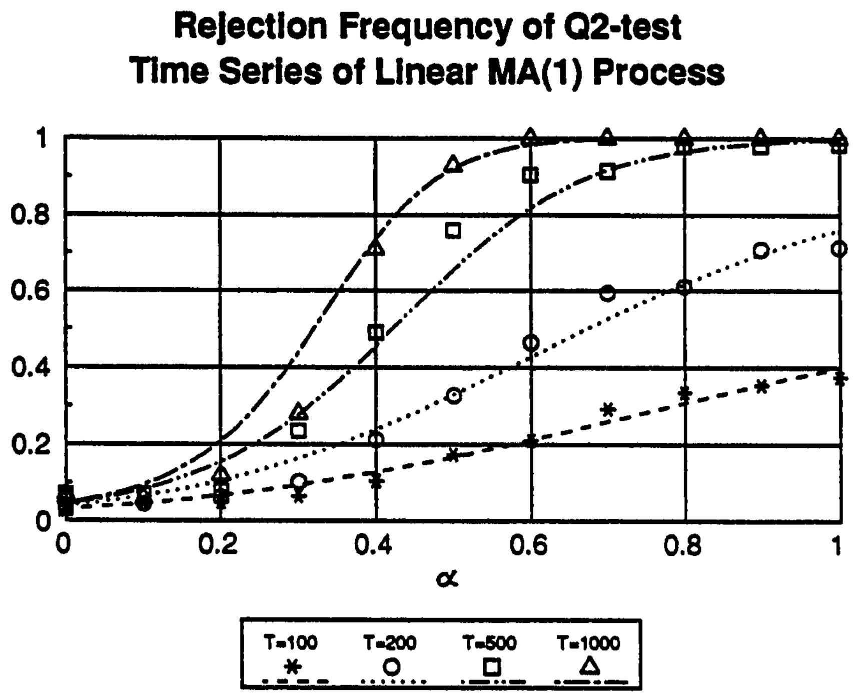

Rejection Frequency of Q2-test
Time Series of Linear MA(1) Process
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100
T=200
T=500
T=1000

Figure 5.19

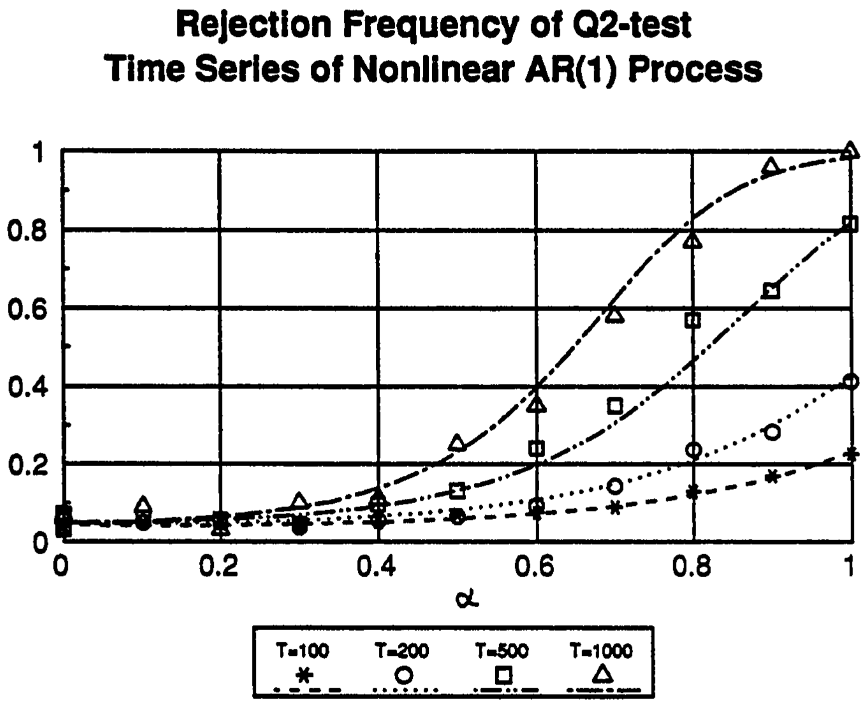

Rejection Frequency of Q2-test
Time Series of Nonlinear AR(1) Process
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100
T=200
T=500
T=1000

Figure 5.20

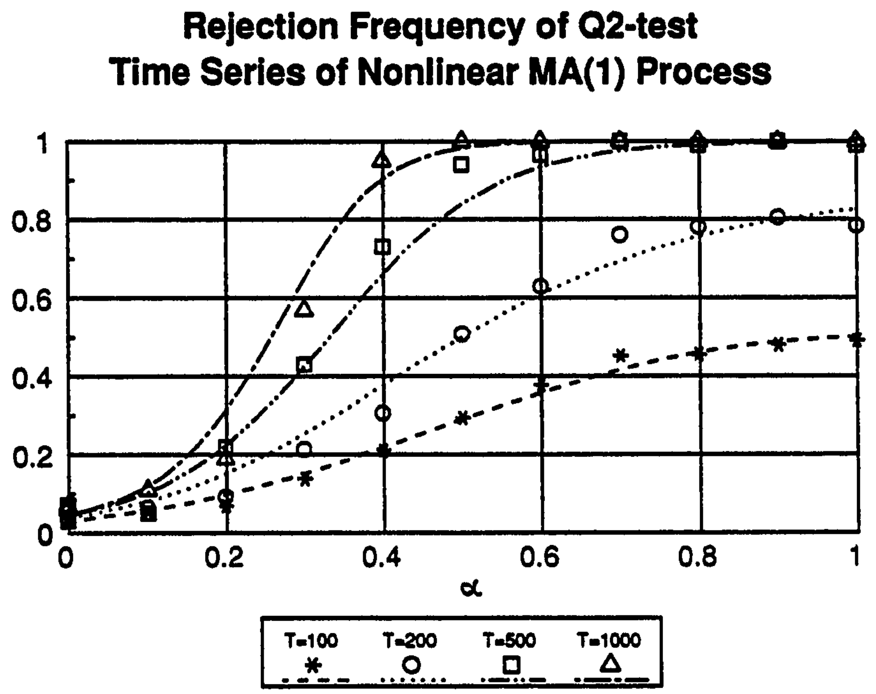

Rejection Frequency of Q2-test
Time Series of Nonlinear MA(1) Process
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
α
T=100
T=200
T=500
T=1000

Figure 5.21

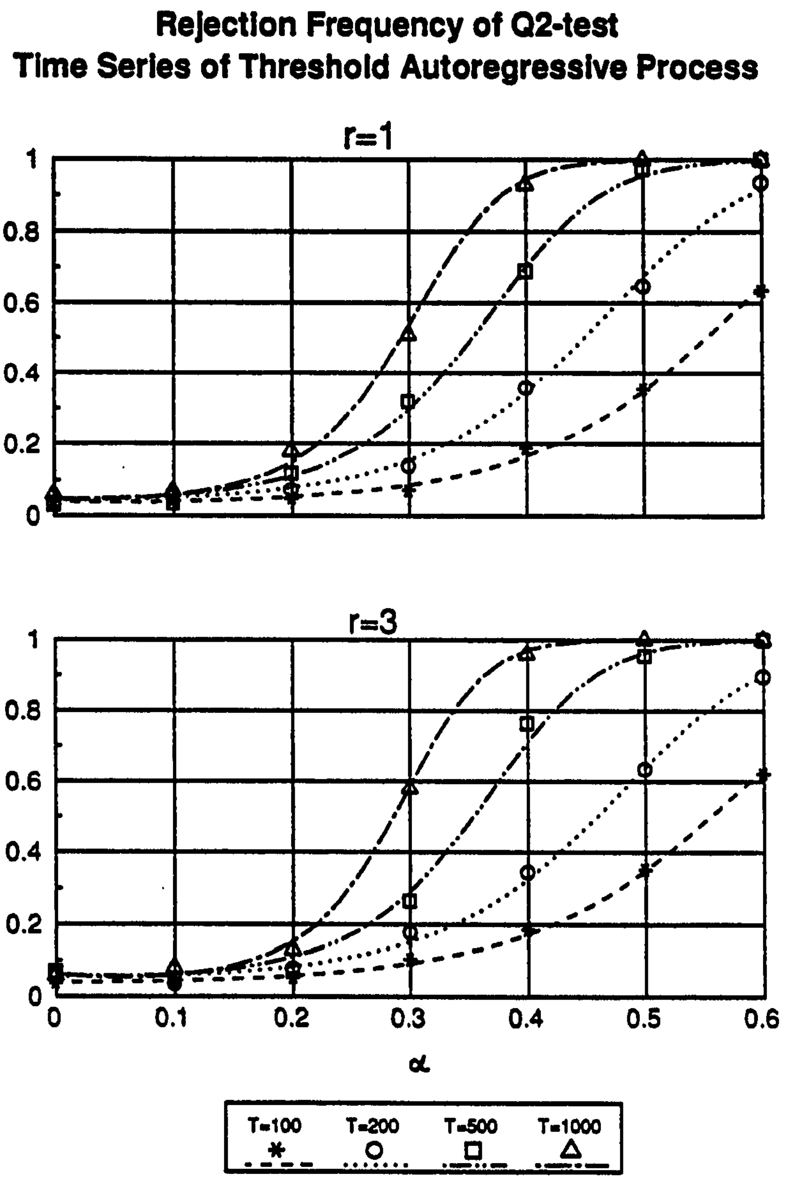
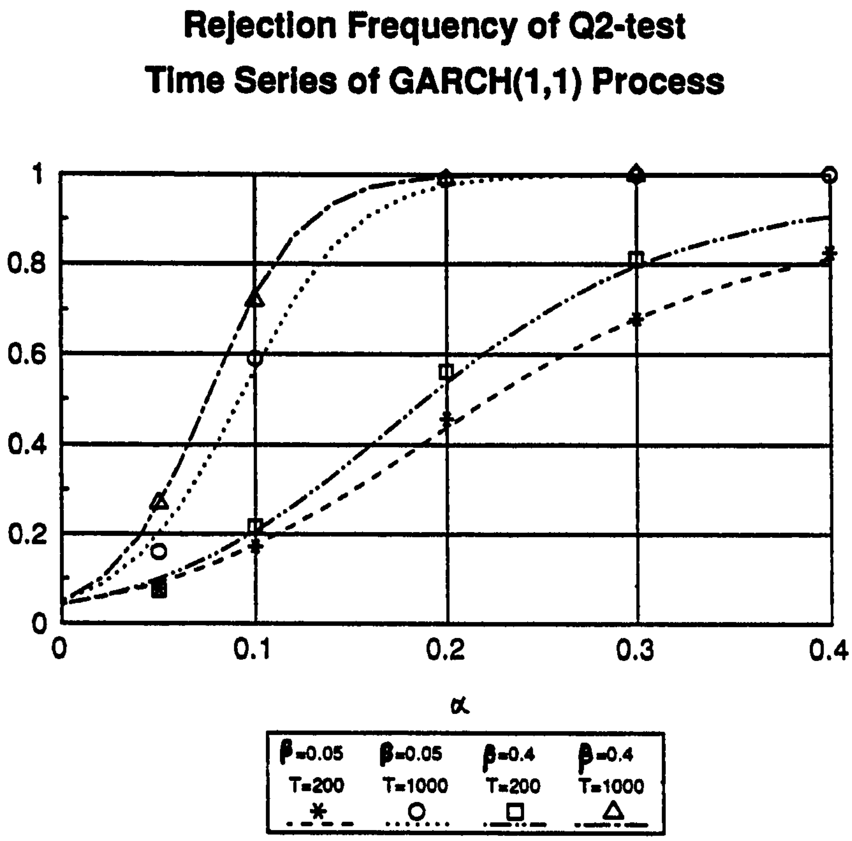

Rejection Frequency of Q2-test
Time Series of Threshold Autoregressive Process
r=1
r=3
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
α
T=100
T=200
T=500
T=1000

Figure 5.22


Rejection Frequency of Q2-test
Time Series of GARCH(1,1) Process
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
α
β=0.05 T=200
β=0.05 T=1000
β=0.4 T=200
β=0.4 T=1000

## 5.7 Comparison of Tests

Compareson of the three test statistics shows that the sizes of the tests under the null hypothesis are close to their asymptotic values for the IID time series we studied (see Figures 5.4, 5.5, 5.6, Tables 5.4, 5.6). For time series of AR(1) process and time series of linear MA(1) process, while the size of the TAR-F test under null hypothesis is close to the theoretical value, the BDS statistic and the $Q^2$ statistic can reject the null hypothesis with large frequency (see Table 5.4, Figures 5.7, 5.8, 5.17, 5.18). For both the time series of linear AR(1) process and the time series of linear MA(1) process, the BDS statistic gives larger rejection frequency than the $Q^2$ statistic does for the 4 sample sizes and all values of the coefficient considered.

For the time series of nonlinear AR(1) process, all three test statistics can reject the null hypothesis with large frequency (see Figures 5.9, 5.13, and 5.19). Among the three test statistics, the TAR-F statistic has largest rejection frequency when the nonlinear AR coefficient is not so large, $\alpha<0.7$. But when the nonlinear AR coefficient is large, say $\alpha>0.7$, the TAR-F statistic has smaller rejection frequency than the BDS statistic and the $Q^2$ statistic do. And in the region of $\alpha>0.7$, the BDS statistic has the highest rejection frequency, especially when the sample size is small, say $T<200$. The TAR-F statistic also indicates the nonlinearity is in the form of lag 1.

For the time series of the nonlinear MA(1) process, the TAR-F statistic has highest rejection frequency among the three test statistics (see Figures 5.10, 5.14, and 5.20). This holds for all 4 sample sizes and for all values of the nonlinear MA coefficient considered, with the exception when the sample size is small (T=100 or 200) and the nonlinear MA coefficient is large ($\alpha > 0.7$). The BDS statistic and the $Q^2$ statistic have very comparable rejection frequency at the 4 sample sizes and all values of the nonlinear MA coefficient considered. The TAR-F statistic also indicates the nonlinearity is in the form of lag 1 and lag 2.

For the time series of the threshold autoregressive (TAR) process, the TAR-F statistic also has the highest rejection frequency among the three test statistics, at the 4 sample sizes, at all threshold lags and for all coefficient values of the TAR process considered (see Figures 5.11, 5.15, and 5.21). The BDS statistic has comparable rejection frequency with rejection frequency of the $Q^2$ statistic for the time series of TAR process with threshold lag of 3, but has much lower rejection frequency than the $Q^2$ statistic does for the time series of TAR process with threshold lag of 1. For the time series of TAR process with threshold lag of 1, 2, 3 and 4, the TAR-F statistic shows its ability to identify the threshold lag, which is an useful property for model building of the TAR process.

For the time series of GARCH(1,1) process, the BDS statistic out-performs the other two test statistics (see Figures 5.12, 5.16, and 5.22). The $Q^2$ statistic gives much higher rejection frequency than the TAR-F statistic does, but the BDS statistic does even better. When the first coefficient of the GARCH(1,1) process is small and the second coefficient of the GARCH(1,1) process is large, i.e., when the past value of the time series has little influence on the current variance and the past variance has strong influence on the current variance, the TAR-F statistic and the $Q^2$ statistic give low rejection frequency, but the BDS statistic continues to provide large rejection frequency.

In short, for the time series studied, the TAR-F statistic has the ability to pick up time series of nonlinear process and to identify the lag variable involved in the nonlinear process. The BDS statistic and the $Q^2$ statistic have large rejection frequency for time series of linear process as well as for time series of nonlinear process. With the exception for the time series of GARCH(1,1) process where the BDS statistic has better performance, the TAR-F statistic is preferred for detecting the nonlinear time series.

## 5.8 Summary

In this chapter we studied the finite sample properties of the BDS test, the TAR-F test, and the $Q^2$ test using Monte Carlo experiments. Monte Carlo experiments consisted of nine data generating processes (DGPs) which represent three IID time series, two linear processes, and four nonlinear time series processes. For each DGP we also considered several parameter values of DGP and four sample sizes. The results of Monte Carlo experiments are summarized using response surfaces. These response surfaces show how the variation of the test statistics can be linked to experiment sample size and to variation in the parameters of the DGP. Following is the summary of my Monte Carlo investigation.

For the study of finite sample properties of the BDS test, we first examine the selection of the correlation length L and the embedding dimension M for calculating the BDS statistic. The Monte Carlo experiments showed that the selection of L=1.5 and

M=3 will minimize the standard deviation of the BDS statistic for all DGPs considered. For DGPs of IID time series, the selection of the L=1.5 and M=3 will minimize the deviation of the rejection frequency of the BDS statistic from the size of asymptotic test under the null hypothesis. For DGPs of non-IID time series, the selection of L=1.5 and M=3 will maximize the rejection frequency of the BDS statistic. The results indicated that the BDS test can yield most reliable results at L=1.5 and M=3. The results here are important to researchers because the number of the BDS statistics to be calculated can be cut down and we do not have to look at many BDS statistics calculated and decide which one to use for hypothesis test.

With L=1.5 and M=3, the Monte Carlo experiments showed that the mean, the standard deviation, and the rejection frequency of the BDS statistic with finite sample for DGPs of IID time series are close to their respective values of asymptotic distribution under the null hypothesis. The mean, the standard deviation, and the rejection frequency of the BDS statistic with finite sample for DGPs of linear and of nonlinear time series processes will increase as the parameter values of DGPs increase and/or sample size increases. The BDS test has high power to pick up DGPs of non-IID time series processes. With moderate parameter values of DGPs of non-IID time series, the power of the BDS test reaches unity at sample size of 1000. If the parameter values of DGPs of non-IID time series are larger, the power of the BDS test can reach unity at smaller sample size.

For the TAR-F test, the rejection frequency of the TAR-F statistic with finite sample size for DGPs of IID time series processes and of linear time series processes is close to the size of finite sample test under the null hypothesis. The rejection frequency of the TAR-F statistic with finite sample for DGPs of nonlinear time series processes will increase when the parameter values of the DGPs of nonlinear time series processes increase and/or sample size increases. The TAR-F test has good power to reject the null hypothesis for DGPs of nonlinear time series processes. With moderate parameter values of DGPs of nonlinear time series, the power of the TAR-F reaches unity at sample size of 1000 except for DGP of GARCH process. The Monte Carlo experiment also showed that the TAR-F statistic can be used for identifying the threshold lag of a TAR process. This results is very useful for model building of TAR model. And it is used for modeling futures prices in Chapter.

For the $Q^2$ test, the rejection frequency of the $Q^2$ statistic with finite sample for DGPs of IID time series processes is close to in size to that of the asymptotic test under the null hypothesis. The rejection frequency of the $Q^2$ statistic with finite sample for DGPs of linear and of nonlinear time series processes will increase when the parameter values of the DGPs increase and/or the sample size increases. The $Q^2$ test has good power against DGPs of linear and of nonlinear time series processes. With moderate parameter values of DGPs of non-IID time series, the power of the $Q^2$ test also reaches unity at sample size of 1000.

Comparing the finite sample performance of the three tests, we found that the BDS statistic has higher power than the $Q^2$ statistic does against the DGPs of linear time series processes. The TAR-F statistic has highest power among the three against DGPs of nonlinear time series processes except the DGP of the GARCH process, where the

BDS statistic has the highest power. For all three tests, their finite sample rejection frequency under the null hypothesis is close to that of the asymptotic test. Under the alternative hypothesis, where the parameter values of the DGP represent the departure from the null hypothesis, the power of the tests reaches unity at moderate parameter values of the DGP and sample size of 1000. If the parameter values of the DGP are larger, the power of the tests can reach unity at smaller sample size. If the sample size is larger, the power of the tests can reach unity at smaller parameter values of DGP. However, if either one or both the parameter values of DGP and the sample size are small, the power of these tests will be much less than unity.

# CHAPTER 6

# MODELING PRICE DYNAMICS IN FUTURES MARKETS

## 6.1 Introduction

In the last two chapters we reviewed two nonlinear time series models and evaluated the finite sample properties of three new statistical tests for detecting serial dependence in time-series data. These models and statistical tests provide us with information that can be used for modeling economic and financial time-series data. In this chapter we use findings reported in Chapter 5 to construct two econometric nonlinear time-series models of price movements in futures markets.

The future markets are special types of financial markets. Studies of price movements in futures markets are somewhat limited in comparison to those relating to the stock markets. Most of the past studies of futures markets relied on simple statistical methods such as the analysis of distribution and the autocorrelation of price changes. More recently, McCurdy and Morgan (1988), Baillie and Myers (1989), Yang and Brorsen (1993) used the GARCH model to study the price changes of some futures.

In this chapter five futures of S&P 500 index, Crude Oil, Japanese Yen, Deutsche Mark, and Eurodollar are studied. In the following sections, the data of futures prices are discussed. Then the data are tested for nonstationary to determine if we should use log-price changes series for studying the price movements for these futures. Before we apply any nonlinear model, the data are tested for serial dependence and for nonlinear dependence. A linear model is used to fit the data and the residuals of the linear model are tested for serial dependence. The linear model used is a linear AR model with the days-of-the-week effect and the holiday effect in the conditional mean. The AR order of the model is determined by Akaike information criterion.

Three statistical tests used in this chapter for testing serial dependence and model adequacy for futures prices are the BDS test, the TAR-F test, and the $Q^2$ test. These three tests are studied in Chapter 5 using Monte Carlo experiments. The results in Chapter 5 showed that these three tests are quite effective at detecting serial dependence. Therefore we can use these three tests in this chapter for detecting and modeling nonlinear serial dependence in futures prices.

Two nonlinear time-series models are applied to price changes of the five futures. The first one, the GARCH model, accounts for serial dependence in conditional variance and thus allows a time-varying variance. The GARCH model specified also allows different variance for different days of the week and holidays to accommodate the days-of-the-week effect and the holiday effect in variance. The GARCH model is estimated using maximum likelihood method.

The second model is the TAR model. It is a piece-wise linear AR model, because when a lagged value of the time series falls into different threshold regions the coefficients and variances of the linear process will take different values. The lag variable and the threshold region of the TAR model can be identified using an arranged recursive regression (see Section 4.5). Once the threshold lag and threshold regions are identified, the TAR model is estimated using the ordinary least squares.

The last section of this chapter summarizes the results. A comparison with previous studies is also provided. Finally the relation of the results to the theoretical financial economic models reviewed in Chapter 3 is discussed.

## 6.2 The Data

The futures prices studied here are the daily closing prices. Let $y_t$ be the futures closing price on day t, then we can calculate the log-price change[1] on day t as:

$$x_t = \log(y_t) - \log(y_{t-1}) ,$$

where log() denotes the natural logarithm function. Here $x_t$ also can be considered as the rate of price change on day t. The log-price changes are multiplied by 100 to be expressed in percentage terms.

Every futures contract has a maturity date. Futures contract names are derived from the particular month in which they mature, eg., the March contract or June contract. An asset can have futures contracts for every month, or every 3 months, or

[1] For futures, people use the term 'log price change' rather than the term 'return' because buying a futures does not require the payment in full unless one receive the delivery of the asset underlying the futures.

even every 12 months. At any time, several futures contracts of an asset with different maturity date can be traded. The futures contract which has the earliest maturity date is called a nearby contract, and the futures contract which has second earliest maturity date is called a first-defer contract. For a futures contract, at the end of the maturity month the price of the futures contract is no longer available. So if we need the log-price changes for several years, log-price changes of the futures from many contracts have to be used.

Taylor (1986) combines the log-price changes from many futures contracts by using the log-price changes of the nearby contract up to the month before the maturity month of the contract. And during the month when the nearby contract matures, the log-price changes of the first-defer contract is used. For example, the S&P 500 index futures has contracts for the months of March, June, September, and December. In January and February the log-price changes of the March contract are used. In March, April, and May the log-price changes of the June contract are used.

Because trading of a futures contract can be continued to the end of the contract's maturity month, Yang and Brorsen (1993) combine the log-price changes from many contracts by using the log-price changes of the nearby contract up to the third Tuesday of its maturity month. For the example of S&P 500 futures, in January, February, and up to third Tuesday of March, the log-price changes of the March contract are used. After third Tuesday of March and up to third Tuesday of June, the log-price changes of the June contract are used.

In this dissertation, we combine the log-price changes from many contracts based on contract trading-volume. In general the contract which is being traded more actively and has larger trading volume also draws larger number of market participants and reveals more price information than a less active contract does. Thus we use the log-price changes of the contract which has the largest trading volume. For most futures, the nearby contract draws the most trading activity until a certain date in its maturity month. The date when the nearby contract starts to have lower trading volume than the first-defer contract is necessarily the first day nor the third Tuesday of the maturity month of the nearby contract. For example, the nearby contract of the S&P 500 futures can start to have lower trading volume ranging from the 7th day to the 17th day of its maturity month. The trading volume of the nearby contract can be as low as one fifth of the trading volume of the first-defer contract on the third Tuesday of the maturity month of the nearby contract.

Table 6.1 lists the futures studied in this chapter. The data cover nearly 10 years from 1984 to 1993 and consist of more than 2000 observations of daily log-price changes. This sample size is typical of those used in studies of futures markets.

Table 6.1
Futures and Sample Period

| Futures | Sample Period | Contract Months | Sample Size |
|---|---|---|---|
| S&P 500 | 1/84 - 8/93 | Mar May Sep Dec | 2444 |
| Crude Oil | 1/84 - 6/93 | Every month | 2385 |
| J. Yen | 1/84 - 6/93 | Mar May Sep Dec | 2401 |
| D. Mark | 1/84 - 6/93 | Mar May Sep Dec | 2401 |
| Eurodollar | 1/84 - 6/93 | Mar May Sep Dec | 2401 |

## 6.3 Univariate Time Series Properties of the Data

In this section, the univariate time series properties of the data are examined. First we test the nonstationary properties of the data, and determine the procedures needed to reduce the nonstationarity in our analysis. Then we apply the time series statistical test to identify the specific form of serial dependence. Finally we estimate a linear time series model and test whether the linear time series model can account for the serial dependence in the data. In the next two section we fit two nonlinear time series models to the data.

### 6.3.1 Testing Nonstationary Properties of the Data

Before we attempt to model the time series of futures prices, we need to check whether the time series data is stationary. It is generally observed that many economic and financial time series data are nonstationary even when they are measured in real terms. When the dependent variable and the independent variable both are nonstationary, we can have a spurious regression where the variances of the estimated parameters are usually underestimated.[2]

Nelson and Plosser (1982) suggest that there are two kinds of nonstationary time series data: trend stationary and difference stationary. If the time series data is difference stationary, then we have to take the difference of the time series to reduce the nonstationary time series to stationary time series before specifying a model. The

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[2] For detail see Davidson and MacKinnon (1993).

techniques used for identifying two types of nonstationary time series data are unit root tests. The concept of unit root tests is to test whether the polynomial of the lag operator in the AR process of the variable has a unit root. For example, in the following regression equation

$$y_t - y_{t-1} = \beta_0 + \beta_1 t + (\alpha-1)y_{t-1} + u_t \ , \tag{6.3.1}$$

when $\alpha=1$, then the polynomial of the AR lag operator has a root equal to 1 (i.e, a unit root). If the test rejects the unit root, then the time series $y_t$ is not difference stationary. Therefore we can model $y_t$ directly without using its difference. If, however, the test fails to reject the unit root, we have take difference of $y_t$ to reduce the nonstationary time series to a stationary time series before we specify a model for $y_t$. A simple and popular test for unit root is the Dickey-Fuller test developed by Fuller (1976) and Dickey and Fuller (1979). The statistic in a Dickey-Fuller test is the t value (call $\tau$ statistic) of the coefficient ($\alpha$-1), and its critical values are provided by Dickey and Fuller (1979).[3] The augmented Dickey-Fuller test (ADF test) also include the lagged terms of $y_t - y_{t-1}$ in the regression equation.

In this section we report results of a unit root test in log-prices of the futures. If the unit root is accepted, then we need to use the first difference of the log-prices (i.e., the log-price change) in the model building. Furthermore, we test for unit root in the log-

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[3] The table of the critical values of Dickey-Fuller tests also can be found in books such as Harvey (1989), Davidson and MacKinnon (1993).

price changes to see if we need to use first difference of log-price changes for model building. Thus if $y_t$ is the log-price of futures, denote $y_t$-$y_{t-1}$ as $\Delta y_t$, then the ADF test of unit root in log-price is based on:

$$\Delta y_t = \beta_0 + \beta_1 t + (\alpha-1)y_{t-1} + \Sigma_{i=1,10} \Delta y_{t-i} + u_t \ . \tag{6.3.2}$$

The log-price change of the futures is $x_t = y_t - y_{t-1}$, and its unit root test is based on:

$$\Delta x_t = \beta_0 + \beta_1 t + (\alpha-1)x_{t-1} + \Sigma_{i=1,10} \Delta x_{t-i} + v_t \ . \tag{6.3.3}$$

The results of unit root test on both log-prices and log-price changes of the futures are shown in Table 6.2. The results indicate that the time series of log-prices have unit roots. Thus we need to use first difference of the log-price (i.e., log-price change) to reduce the nonstationary time series to stationary time series for modeling of futures prices. The test rejects the unit root in time series of log-price changes. Therefore we do not need to use the first difference of the log-price changes for modeling futures prices.

Table 6.3 provides summary statistics of the log-price changes of these futures. All the means are not statistically different from zero. The estimated variances of the daily log-price changes are very different from futures to futures. All of the futures daily log-price changes are skewed. The skewness is negative for S&P 500 futures and for Crude Oil futures, but is positive for Japanese Yen futures, for Deutsche Mark futures, and for Eurodollar futures. All the kurtosis is significantly different from that of the normal distribution.

### 6.3.2 Diagnostic Testing and Linear Time Series Models

The next step in analyzing the futures prices is to test for serial dependence in log-price changes. The tests used here are the BDS test, the TAR-F test, the $Q^2$ test, and the Bispectral test (see Section 4.7), as well as the traditional Ljung-Box Q test. The results of these tests on log-price changes for the five futures are presented in Table 6.4.

The Ljung-Box Q statistic shows that there is linear dependence in log-price changes of S&P 500 futures, of Crude Oil futures, and of Eurodollar futures. The $Q^2$ statistic shows that there is serial dependence in the square of the log-price changes of all five futures. The large $Q^2$ statistic can be the results of linear dependence or the results of nonlinear dependence such as the GARCH process. The BDS statistic rejects the null hypothesis of IID for log-price changes of all five futures. The TAR-F statistic rejects the linear time series model for log-price changes of S&P 500 futures, of Crude Oil futures, of Deutsche Mark futures, and Eurodollar futures. The large TAR-F statistic can be the results of nonlinearity in the conditional mean, such as TAR process. The Bispectral test rejects linear time series process for all futures except Crude Oil futures.

Because the BDS statistic rejects the IID null hypothesis for log-price changes of all five futures, we first investigate if the rejection of IID is the results of days-of-the-week effect in the conditional mean and of the linear AR process. So a linear regression model of log-price changes is fitted for each of five futures:

$$x_t = C + \Sigma_{i=1,4}\,\beta_i D_i + \beta_H DH + \Sigma_{j=1,p}\, b_j x_{t-j} + e_t\ , \tag{6.3.4}$$

where $e_t$ follows a stationary process. The $D_i$s are the dummy variables for days-of-the-week, $D_1$ is for Monday, $D_2$ is for Tuesday, $D_3$ is for Wednesday, and $D_4$ is for Thursday, the Friday is the base case. The DH is the dummy variable for holidays. The lag length p of the AR process is determined by the Akaike information criterion. The estimated coefficient of the model and the test results of serial dependence in the residuals are presented in Table 6.5.

For S&P 500 futures, the days-of-the-week effect in the conditional mean is not significant. The holiday effect in conditional mean is not significant either. Some coefficients of the AR process are significant, at lag values of 2, 4, and 5. The residuals of the model pass the Ljung-Box test. But they do not pass neither the $Q^2$ test nor the BDS test. So the log-price changes of the S&P 500 futures can not be adequately modeled by the linear model. This suggest that a nonlinear time series model, such as the GARCH model or the TAR model, may be more appropriate for log-price changes of S&P 500 futures.

For Crude Oil futures, both the days-of-the-week effect and the holiday effect in conditional mean are not significant. The coefficients of AR process are significant at lags of 3, 4, 5, 7, and 8. The residuals of the model can pass the Ljung-Box test. But they can not pass neither the $Q^2$ test nor the BDS test. Therefore we conclude that the log-price changes of the Crude Oil futures has to be modeled by a nonlinear time series model rather than a linear model.

For Japanese Yen futures and Deutsche Mark futures, the days-of-the-week effect and the holiday effect in the conditional mean are not significant. The residuals of the linear model pass the Ljung-Box test. But the residuals do not pass neither the $Q^2$ test nor the BDS test. This result indicates that the log-price changes of Japanese Yen futures and of the Deutsche Mark futures can not be modeled by linear models and they have to be modeled by nonlinear time series models.

For the Eurodollar futures, the days-of-the-week effect in conditional mean is significant for the constant term (Friday) and Monday. The prices tend to raise on Fridays and fall on Mondays. The holiday effect in conditional mean is also significant, and the prices tend to fall during the holidays. The effect of AR process is also significant. The residuals of the model pass the Ljung-Box test, but they failed to pass the $Q^2$ test and the BDS test. The results suggest that linear models are not suitable for the log-price changes of Eurodollar futures, and nonlinear time series models should be used.

Table 6.2
Test of Unit Root in Log-prices and Log-Price Changes

| | ($\alpha$-1) | $\tau$ statistic | ($\alpha$-1) | $\tau$ statistic |
|---|---|---|---|---|
| | Log-prices | | Log-price Changes | |
| S&P 500 | -0.0085 | -2.81 | -1.23 | -15.64[a] |
| Crude Oil | -0.0050 | -2.37 | -1.08 | -14.73[a] |
| J. Yen | -0.0016 | -1.57 | -0.89 | -13.57[a] |
| D. Mark | -0.0020 | -1.44 | -0.94 | -14.10[a] |
| Eurodollar | -0.0021 | -1.75 | -0.89 | -14.02[a] |

[a] Denotes rejection of the null hypothesis of an unit root. The critical value of the $\tau$ statistic at 5% level is -3.45.

Table 6.3
Summary Statistics of Daily Futures Log-Price Changes

| Futures | Maximum | Minimum | Mean | Var. | Skew. | Kurt. |
|---|---|---|---|---|---|---|
| S&P 500 | 17.75 | -33.70 | 0.0277 | 1.654 | -6.690[a] | 210.8[a] |
| Crude Oil | 14.03 | -38.41 | 0.0224 | 5.626 | -1.946[a] | 35.16[a] |
| J. Yen | 5.33 | -4.13 | 0.0251 | 0.453 | 0.405[a] | 4.12[a] |
| D. Mark | 4.83 | -3.31 | 0.0162 | 0.609 | 0.156[a] | 1.90[a] |
| Eurodollar | 0.297 | -0.106 | 0.0015 | 0.00053 | 1.128[a] | 13.76[a] |

[a] Denotes rejection of the null hypothesis of a normal distribution (see Judge et. al., 1988, p 891, for the definition and the asymptotic distribution of skewness and kurtosis).

Table 6.4
Test of Serial Dependence in Futures Log-Price Changes

| Futures | Q(24) | $Q^2$(24) | BDS | TAR-F | Bispectral |
|---|---|---|---|---|---|
| S&P 500 | 109[a] | 207[a] | 8.73[a] | 32.92[a] | 3.15[a] |
| Crude Oil | 76.1[a] | 221[a] | 21.16[a] | 5.27[a] | 1.84 |
| J. Yen | 29.2 | 90.9[a] | 6.97[a] | 0.84 | 3.62[a] |
| D. Mark | 24.3 | 156[a] | 3.09[a] | 2.45[a] | 3.10[a] |
| Eurodollar | 44.6 | 105[a] | 6.12[a] | 3.94[a] | 6.46[a] |

[a] Denotes rejection of the null hypothesis.

The Ljung-Box Q statistic and the McLeod-Li $Q^2$ statistic are calculated at lag 24. The BDS statistic is calculated at L=1.5 and M=3. The TAR-F statistic is calculated at d=3 and p=10, where the results of TAR-F statistic at other threshold lag d are similar. The Bispectral test is calculated at the grid size of 49 and the smooth length of 63.
At 1% significant level, the critical value for Q test and $Q^2$ test is 42.98, the critical value for BDS test and Bispectral test is 2.23, the critical value for TAR-F test is 2.25.

Table 6.5
Estimated Linear Models and Test of Serial Dependence in Residuals

| | S&P 500 | Crude Oil | J. Yen | D. Mark | Eurodollar |
|---|---|---|---|---|---|
| C | -0.044 | -0.055 | 0.0015 | -0.026 | 0.00202 |
| | (0.058) | (0.109) | (0.031) | (0.036) | (0.00105)* |
| $\beta_1$ | 0.082 | 0.085 | 0.027 | 0.058 | -0.0023 |
| | (0.082) | (0.156) | (0.043) | (0.051) | (0.0015) |
| $\beta_2$ | 0.115 | -0.034 | 0.0079 | 0.049 | 0.0021 |
| | (0.081) | (0.153) | (0.043) | (0.050) | (0.015) |
| $\beta_3$ | 0.132 | 0.140 | 0.023 | 0.059 | -0.0010 |
| | (0.081) | (0.152) | (0.043) | (0.050) | (0.0015) |
| $\beta_4$ | 0.058 | 0.209 | 0.068 | 0.056 | -0.0010 |
| | (0.081) | (0.153) | (0.043) | (0.050) | (0.0015) |
| $\beta_H$ | -0.070 | -0.013 | -0.054 | -0.058 | -0.0057* |
| | (0.149) | (0.274) | (0.080) | (0.092) | (0.0027) |
| $b_1$ | -0.016 | 0.032 | | | -0.084* |
| | (0.020) | (0.021) | | | (0.020) |
| $b_2$ | 0.159* | -0.035 | | | |
| | (0.020) | (0.021) | | | |
| $b_3$ | -0.019 | -0.078* | | | |
| | (0.020) | (0.021) | | | |
| $b_4$ | -0.062* | 0.058* | | | |
| | (0.020) | (0.021) | | | |
| $b_5$ | 0.072* | -0.093* | | | |
| | (0.020) | (0.021) | | | |
| $b_6$ | | 0.0004 | | | |
| | | (0.021) | | | |
| $b_7$ | | 0.193* | | | |
| | | (0.021) | | | |
| $b_8$ | | -0.080* | | | |
| | | (0.021) | | | |
| Test of Serial Dependence in Residuals | | | | | |
| Q(24) | 33.3 | 32.0 | 29.1 | 24.9 | 29.6 |
| $Q^2$(24) | 67.0[a] | 231[a] | 90.1[a] | 157[a] | 114[a] |
| BDS | 8.18[a] | 20.07[a] | 7.14[a] | 3.11[a] | 6.21[a] |

[a] Denotes rejection of the null hypothesis. * Significant at 5% level.
Standard errors of the estimated parameters are in parentheses.
See the notes for Table 6.4.

## 6.4 Nonlinear Time Series Model I - GARCH Model

In the last section, we tested and found that the log-price changes of all five futures have serial dependence. The log-price change of each futures are fitted by a linear model (which also includes the days-of-the-week effect as well as the holiday effect in the conditional mean), the residuals of the model do not pass neither the $Q^2$ test nor the BDS test. This indicates that the log-price changes of all five futures have to be modeled by nonlinear models. In this section we examine whether the GARCH model is appropriate for modeling the log-price changes of these five futures.

The GARCH(1,1) model is specified as:

$$x_t = C + \Sigma_{i=1,4}\, \beta_i D_i + \beta_H DH + \Sigma_{j=1,p}\, b_j x_{t-j} + e_t \,,\; e_t \sim N(0,h_t)$$

$$h_t = \alpha_0 + \alpha_1 e^2_{t-1} + \alpha_2\, h_{t-1} + \Sigma_{i=1,4}\, a_i D_i + a_H DH \,. \quad (6.4.1)$$

The $D_i$s dummies for the days-of-the-week effect and the DH is the dummy for holiday effect. In this model, the conditional mean of the log-price changes has AR process, the days-of-the-week effect, and the holiday effect. The lag of the AR process in the conditional mean is determined by the Akaike information criterion in the last section. The conditional variance of log-price changes is an autoregressive process of past sample variance and past conditional variance. The model also has the days-of-the-week effect and the holiday effect in the conditional variance. The GARCH model is estimated by maximum likelihood method. The results of the estimation are presented in Table 6.6.

Our findings based on the GARCH model are as follows. For the conditional mean of log-price changes in the model, the results of the days-of-the-week and holiday effect produce little change from results of the linear model estimated in last section. However, fewer coefficients of the AR process remain significant for S&P 500 futures and the Crude Oil futures. This implies that if we omit the effect of a GARCH process, we may incorrectly make inference about the existence of the AR process in the conditional mean of log-price changes.

For the conditional variance of log-price changes in the model, the effect of sample variance and the effect of conditional variance in last period is significant. But the effect of past sample variance is less than the effect of past conditional variance. All futures have significant days-of-the-week and holiday effect in variance.[4] For Crude Oil futures the variance on Mondays is large but the variance is small for rest of the week. For Eurodollar futures the opposite is true. For all the futures except Eurodollar futures, the variance in mid-week is smaller than the variance on Mondays and Fridays. For all five futures, the variance over the holidays are significantly large, especially for the Crude oil futures. The results are in support of the calendar time hypothesis than the trading time hypothesis.

The test results for serial dependence in the standardized residual of the GARCH model are also presented in Table 6.6. For Japanese Yen, Deutsche Mark, and

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[4]The TSP program used for estimation of GARCH reported that the days-of-the-week effect on Monday can not be estimated for S&P 500 futures due to singularity of the data. This is probably caused by the large decline of stock prices on Black Monday of 1987.

Eurodollar futures, the standardized residuals pass the Ljung-Box, $Q^2$, and BDS test. The results imply that the standardized residuals of the GARCH model IID. Therefore we conclude that the GARCH process can adequately model the log-price changes of these three futures. The standardized residuals of the S&P 500 futures and of the Crude Oil futures pass the Ljung-Box test and the $Q^2$ test. But they do not pass the BDS test. Therefore we have to look for other models to analyze the log-price changes of S&P 500 futures and Crude oil futures.

Yang and Brorsen (1993) also found the serial dependence in the standardized residuals of the GARCH model for the log-price changes of some futures they studied. They suggested the use of a higher order GARCH model.[5] For S&P 500 and Crude Oil futures, because the value of $Q^2$ statistics for the standardized residuals of the GARCH model is low, this implies that the standardized residuals are unlikely to have autocorrelation in their squared values. And the BDS test's rejection of IID for standardized residuals of the GARCH model is probably not the results of higher order autoregressive conditional heteroskedasticity. Therefore using of higher order GARCH model will not be effective at removing the serial dependence in the standardized residuals of the model. Based on the large TAR-F statistic from the log-price changes of S&P 500 futures and of Crude Oil futures calculated in the last section, we apply the TAR model to study the price changes of these 2 futures in the next section.

[5] Brock, Hsieh, and LeBaron (1991) also found the GARCH model can not model all the nonlinearities in the log price change of most of foreign currencies they studied. And their modeling of the price changes in foreign currencies was stopped at the GARCH model.

**Table 6.6**
**Estimated GARCH(1,1) Models and Test of Serial Dependence in Residuals**

| | S&P 500 | Crude Oil | J. Yen | D. Mark | Eurodollar |
|---|---|---|---|---|---|
| c | 0.017 | 0.032 | -0.013 | -0.015 | 0.0022* |
| | (0.045) | (0.058) | (0.031) | (0.035) | (0.0010) |
| $\beta_1$ | 0.269* | 0.077 | 0.032 | 0.039 | -0.0033* |
| | (0.063) | (0.087) | (0.046) | (0.050) | (0.0012) |
| $\beta_2$ | 0.098 | -0.124 | 0.034 | 0.033 | -0.00018 |
| | (0.061) | (0.084) | (0.043) | (0.049) | (0.00122) |
| $\beta_3$ | 0.075 | -0.023 | 0.028 | 0.051 | -0.00072 |
| | (0.056) | (0.081) | (0.039) | (0.045) | (0.00122) |
| $\beta_4$ | -0.017 | 0.152* | 0.055 | 0.050 | -0.00053 |
| | (0.065) | (0.075) | (0.042) | (0.048) | (0.00134) |
| $\beta_H$ | -0.241* | 0.032 | -0.127 | -0.072 | -0.0037 |
| | (0.108) | (0.148) | (0.071) | (0.095) | (0.0024) |
| $b_1$ | -0.0017 | 0.013 | | | -0.078* |
| | (0.0019) | (0.017) | | | (0.024) |
| $b_2$ | -0.0068 | 0.0004 | | | |
| | (0.018) | (0.017) | | | |
| $b_3$ | -0.0019 | 0.057* | | | |
| | (0.020) | (0.016) | | | |
| $b_4$ | 0.0088 | 0.004 | | | |
| | (0.019) | (0.017) | | | |
| $b_5$ | 0.017 | 0.027 | | | |
| | (0.019) | (0.016) | | | |
| $b_6$ | | -0.031 | | | |
| | | (0.017) | | | |
| $b_7$ | | 0.036* | | | |
| | | (0.017) | | | |
| $b_8$ | | -0.010 | | | |
| | | (0.018) | | | |
| $\alpha_0$ | 0.083* | 0.019 | 0.119* | 0.142* | 0.00027 |
| | (0.026) | (0.086) | (0.019) | (0.036) | (0.0002) |
| $\alpha_1$ | 0.0196* | 0.043* | 0.089* | 0.0604* | 0.114* |
| | (0.0011) | (0.0032) | (0.011) | (0.0087) | (0.006) |
| $\alpha_2$ | 0.864* | 0.915* | 0.828* | 0.887* | 0.837* |
| | (0.014) | (0.0065) | (0.019) | (0.017) | (0.0092) |
| $a_1$ | | 0.819* | -0.056 | -0.097 | -0.000531* |
| | | (0.149) | (0.036) | (0.062) | (0.000036) |
| $a_2$ | -0.054 | -0.555* | -0.129* | -0.215* | -0.00026* |
| | (0.068) | (0.150) | (0.029) | (0.056) | (0.000026) |
| $a_3$ | -0.178* | -0.079 | -0.175* | -0.215* | -0.00028* |
| | (0.058) | (0.107) | (0.026) | (0.048) | (0.000024) |
| $a_4$ | 0.290* | -0.013 | -0.0416* | -0.053 | -0.00017* |
| | (0.061) | (0.155) | (0.0030) | (0.052) | (0.000028) |
| $a_H$ | 0.280* | 0.938* | 0.220* | 0.220* | 0.00021* |
| | (0.068) | (0.136) | (0.049) | (0.049) | (0.000023) |
| Test of Serial Dependence in Standardized Residuals | | | | | |
| Q(24) | 22.0 | 35.8 | 22.5 | 22.5 | 28.4 |
| $Q^2$(24) | 7.68 | 35.5 | 14.6 | 20.3 | 18.1 |
| BDS | 1.78[a] | 4.77[a] | 0.76 | -1.55 | -2.85 |

[a] Denotes rejection of the null hypothesis. * Significant at 5% level.
Standard errors of the estimated parameters are in parentheses.
See the notes for Table 6.4. At 1% significant level, the critical value of the BDS test on the standardized residuals of GARCH model is 1.59 (see Brock et. al., 1991, p 279).

## 6.5 Nonlinear Time Series Model II - TAR Model

Although the log-price changes of three futures considered can be adequately modeled by the GARCH process, the log-price changes of S&P 500 futures and of Crude Oil futures are not modeled well by the GARCH process because their standardized residuals do not pass the BDS test. We also recall that in Section 6.3 it was reported that the log-price changes of S&P 500 futures and of Crude Oil futures has high values of TAR-F statistic. This finding suggested the existence of a TAR process in the data. In this section we apply the TAR model to the log-price changes of S&P 500 futures and of Crude Oil futures using the modeling procedures of Tsay (1989).

The first step to model the TAR process is to determine the threshold lag of the TAR process. Here we calculate the TAR-F statistic of the log-price changes at threshold lags of 1 to 5. The AR order used in the calculation are 5 for S&P 500 futures and 8 for Crude Oil futures, where the AR orders are determined in Section 6.3 by using Akaike information criterion. The results of the TAR-F statistic for S&P 500 futures and for Crude Oil futures are presented in Table 6.7.

From Table 6.7, we can see that the TAR-F statistic is very large at threshold lags of 1 to 5 for log-price changes of S&P 500 futures. The TAR-F statistic at threshold lag of 1 is the highest. Thus we pick the threshold lag of 1 for the TAR model of log-price changes of S&P 500 futures. For log-price changes of Crude Oil futures, the TAR-F statistic is also high for all 5 threshold lags considered. But at the threshold lag of 2, the TAR-F statistic is substantially higher than the TAR-F statistic at other threshold lags. Therefore a threshold lag of 2 is used for estimating the TAR model for log-price changes of Crude Oil futures.

For S&P 500 futures, the arranged recursive regression of the log-price changes is performed with the threshold lag of 1 and the AR order of 5. The scatter-plot of the t-values of the AR coefficients is presented in Figure 6.1. The scatter-plot of the t-values of the AR coefficients is used for determining the threshold values of the model. The places where the t-values have big change are the location of the threshold values. We can see that the t-values of the coefficient has large changes at -0.62 and 1.15. These two values are identified as the threshold values of the model, and the following TAR model is specified:

$$x_t = C^1 + \Sigma_{i=1,4}\, \beta^1_i D_i + \beta^1_H DH + \Sigma_{j=1,5}\, b^1_j x_{t-j} + e^1_t\ ,\ x_{t-1} < -0.62\ ,$$

$$= C^2 + \Sigma_{i=1,4}\, \beta^2_i D_i + \beta^2_H DH + \Sigma_{j=1,5}\, b^2_j x_{t-j} + e^2_t\ ,\ -0.62 \leq x_{t-1} < 1.15\ ,$$

$$= C^3 + \Sigma_{i=1,4}\, \beta^3_i D_i + \beta^3_H DH + \Sigma_{j=1,5}\, b^3_j x_{t-j} + e^3_t\ ,\ x_{t-1} \geq 1.15\ ,$$

where $D_i$s are the dummy variables for days-of-the-week, DH is the dummy variable for holidays, and $e^i_t$'s are IID with zero mean and variance $\sigma^{i2}$. The results of the estimated TAR model for the log-price changes of S&P 500 futures are presented in Table 6.8.

For log-price changes of S&P 500 futures, the results of the estimated TAR model show that in the threshold region I, when the price of last period declined by more than 0.62 percent, the days-of-the-week effect and the holiday effect in conditional mean are not significant. But two of the AR coefficients are significant, which are -0.10 at lag 1 and 0.28 at lag 5. In the threshold region II, when the price changes of last period is between -0.62 and 1.15 percent, the AR coefficients are not significant and the days-of-the-week effect in conditional mean becomes significant for Mondays and for Wednesdays. The holiday effect in conditional mean is still not significant. In the threshold region III, when the price in last period raised by 1.15 percent or greater, the days-of-the-week effect is not significant. The three AR coefficients are significant at lag 2, 4, and 5. Except at lag 1, all other 4 AR coefficients have negative sign where in threshold region I they have positive sign. The results shows that for S&P 500 futures, when the price changes in last period falls into different threshold regions, the price changes in the current period will follow different linear process because in different threshold regions the coefficients and residual variance of the linear process are different.

The test results for serial dependence in the standardized residuals of the TAR model are also presented in Table 6.8. The standardized residuals pass the Ljung-Box Q test, the $Q^2$ test, and the BDS test. These results confirm that the standardized residuals are IID and the log-price changes of S&P 500 futures can be modeled quite well by a TAR process.

For Crude Oil futures, the arranged recursive regression of the log-price changes is conducted with threshold lag of 2 and an AR order of 8. The scatter-plot of the t-values of the AR coefficients for log-price changes of Crude Oil futures is presented in Figure 6.2. We can observe that the t-values start to decline in absolute value at 0.21. Base on this, the following equation is applied to model the log-price changes of Crude Oil futures:

$$x_t = C^1 + \Sigma_{i=1,4}\, \beta^1_i D_i + \beta^1_H DH + \Sigma_{j=1,8}\, b^1_j x_{t-j} + e^1_t\ ,\ x_{t-2} < 0.21\ ,$$

$$= C^2 + \Sigma_{i=1,4}\, \beta^2_i D_i + \beta^2_H DH + \Sigma_{j=1,8}\, b^2_j x_{t-j} + e^2_t\ ,\ x_{t-2} \geq 0.21\ ,$$

again the $D_i$s are the dummy variables for days-of-the-week, DH is the dummy variable for holiday, and $e^i_t$'s are IID with zero mean and variance $\sigma^{i2}$. The results of the estimated TAR model for the log-price changes of Crude Oil futures are also presented in Table 6.8.

For Crude Oil futures, in the threshold region I, when the price change from 2 periods ago is less than 0.21 percent, the AR coefficients are significant at lag of 3, 4, 5, 7, and 8. These coefficients either have opposite sign or have large difference in value than their counter part in region II. We also see that in region II, when the price change from 2 periods ago is 0.21 percent or greater, the AR coefficient at lag 1, 5, and 7 are significant. The coefficients at lag 1 and 7 have opposite sign of the coefficients at lag 1 and 7 in region I. In region II the AR coefficient at lag 5 is less than half in magnitude than the coefficient at lag 5 in region I. In both regions, the days-of-the-week effect and the holiday effect in conditional mean are not significant.

The test results for serial dependence in the standardized residuals of the TAR model for log-price changes of Crude Oil futures are presented in Table 6.8. The standardized residuals pass the Ljung-Box Q test, but they do not pass either the $Q^2$ test nor the BDS test. Because the large value of $Q^2$ statistic, we conclude that the failure of the standardized residuals to pass these test is due to the serial dependence in conditional variance.

To deal with the serial dependence in the standardized residuals of TAR model of log-price changes of Crude Oil futures, we consider the following. First, the estimation of TAR model indicates that when the threshold variable $x_{t-2}$ falls into two different regions, the log-price changes will follow to different processes. Second, the $Q^2$ statistic of the TAR model's standardized residuals is very large, which indicates the serial dependence in the conditional variance. Third, the estimation of the GARCH model for log-price changes of Crude Oil futures in last section showed the strong serial dependence in conditional variance. Therefore we try a combined TAR-GARCH model in order to account both the conditional variance change and conditional mean change in the log-price changes of Crude Oil futures. The model is specified as:

$$x_t = C + \Sigma_{i=1,4}\,\beta_i D_i + \beta_H DH + \Sigma_{j=1,8}\, b^2_j x_{t-j} + \Sigma_{j=1,8}\, b^1_j D\, x_{t-j} + e_t\,,\ e_t \sim N(0,h_t)$$

$$h_t = \alpha_0 + \alpha_1 e^2_{t-1} + \alpha_2\, h_{t-1} + \Sigma_{i=1,4}\, a_i D_i + a_H DH\ ,$$

$$D = 1 \text{ if } x_{t-2} < 0.21\ , \text{ and } 0 \text{ elsewhere,}$$

where $D_i$s are the dummy variables for days-of-the-week, DH is the dummy variable for holidays. The introducing of the dummy variable D here allows the AR coefficients to be different in different threshold region. The results of the estimated model for the log-price changes of Crude Oil futures are presented in Table 6.9.

From the results reported in Table 6.9, we see that the days-of-the-week effect and the holiday effect in conditional mean are insignificant. The AR coefficient with dummy variable at lag 5 is significant. This shows that the conditional mean of log-price changes follows a different AR process in different threshold regions. The serial dependence in conditional variance is strong, and the days-of-the-week effect and the holiday effect in variance is significant, just as we observed in the GARCH model discussed in the last section.

The test results for serial dependence in the standardized residuals of the model are presented in Table 6.9 as well. The standardized residuals pass the Ljung-Box test, the $Q^2$ test, and the BDS test. Therefore we conclude that the combined TAR-GARCH model specified can adequately describe the log-price changes of Crude Oil futures.

Table 6.7
TAR-F Statistic for Log-price Changes

| Threshold Lag | S&P 500 | Crude Oil |
|---|---|---|
| 1 | 39.12[a] | 4.24[a] |
| 2 | 33.83[a] | 13.91[a] |
| 3 | 35.82[a] | 6.00[a] |
| 4 | 35.52[a] | 2.99[a] |
| 5 | 24.56[a] | 6.67[a] |

[a] Denotes the rejection of null hypothesis.

Note: The AR order used for calculating TAR-F statistic are respectively 5 and 8 for S&P 500 futures and Crude Oil futures. The selection of the AR orders is based on the results of Akaike information criterion in Section 6.3.

**Table 6.8**
**Estimated TAR Models and Test of Serial Dependence in Residuals**

| | S&P 500 Region I $(x_{t-1}<-0.62)$ | S&P 500 Region II $(-0.62 \le x_{t-1}<1.15)$ | S&P 500 Region III $(x_{t-1} \ge 1.15)$ | Crude Oil Region I $(x_{t-2}<0.21)$ | Crude Oil Region II $(x_{t-2} \ge 0.21)$ |
|---|---|---|---|---|---|
| C | -0.237 | -0.053 | 0.360 | -0.108 | -0.045 |
| | (0.212) | (0.050) | (0.286) | (0.152) | (0.175) |
| $\beta^i_1$ | -0.195 | 0.142 | 0.590 | 0.070 | 0.029 |
| | (0.275) | (0.071) | (0.313) | (0.213) | (0.225) |
| $\beta^i_2$ | 0.396 | 0.078 | 0.047 | 0.083 | -0.342 |
| | (0.287) | (0.070) | (0.287) | (0.203) | (0.233) |
| $\beta^i_3$ | 0.143 | 0.134 | -0.194 | 0.056 | 0.118 |
| | (0.278) | (0.071) | (0.282) | (0.205) | (0.225) |
| $\beta^i_4$ | 0.182 | 0.045 | -0.032 | 0.264 | 0.065 |
| | (0.286) | (0.069) | (0.300) | (0.202) | (0.232) |
| $\beta^i_H$ | 0.099 | -0.134 | -0.162 | 0.165 | -0.005 |
| | (0.526) | (0.125) | (0.642) | (0.357) | (0.423) |
| $b^i_1$ | -0.100$^*$ | -0.011 | -0.192$^*$ | -0.046 | 0.102$^*$ |
| | (0.051) | (0.051) | (0.102) | (0.029) | (0.029) |
| $b^i_2$ | 0.137 | -0.039 | -0.395$^*$ | -0.059 | 0.014 |
| | (0.080) | (0.024) | (0.035) | (0.032) | (0.043) |
| $b^i_3$ | 0.132 | -0.004 | -0.021 | -0.127$^*$ | -0.012 |
| | (0.085) | (0.023) | (0.051) | (0.027) | (0.031) |
| $b^i_4$ | 0.050 | -0.018 | -0.348$^*$ | 0.100$^*$ | 0.032 |
| | (0.050) | (0.024) | (0.076) | (0.031) | (0.027) |
| $b^i_5$ | 0.276$^*$ | -0.024 | -0.194$^*$ | -0.138$^*$ | -0.057$^*$ |
| | (0.059) | (0.024) | (0.050) | (0.032) | (0.026) |
| $b^i_6$ | | | | -0.032 | 0.026 |
| | | | | (0.029) | (0.029) |
| $b^i_7$ | | | | 0.086$^*$ | -0.065$^*$ |
| | | | | (0.027) | (0.031) |
| $b^i_8$ | | | | -0.122$^*$ | -0.012 |
| | | | | (0.027) | (0.032) |
| $\sigma^i$ | 1.91 | 0.92 | 1.39 | 2.33 | 2.34 |
| **Test of Serial Dependence in Standardized Residuals** | | | | | |
| Q | | 17.9 | | 25.6 | |
| $Q^2$ | | 24.4 | | 248$^a$ | |
| BDS | | -1.00 | | 19.59$^a$ | |

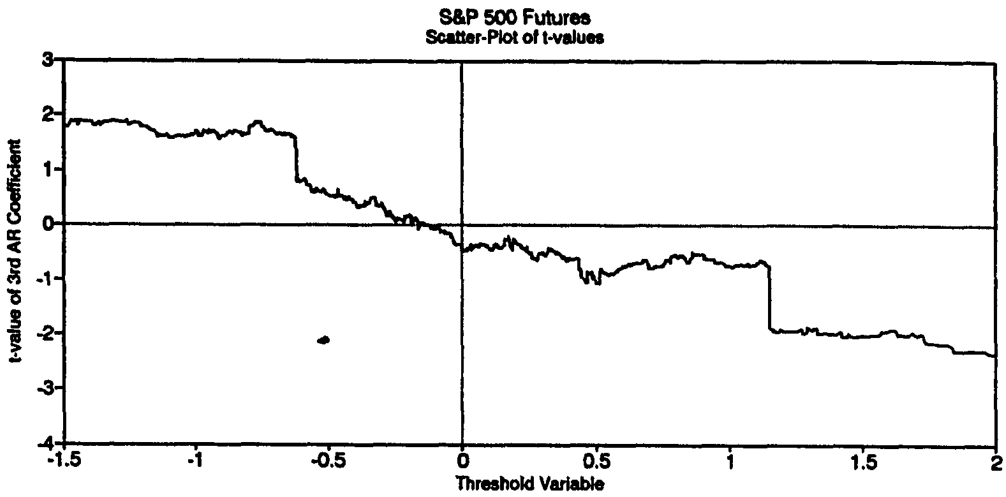
$^a$ Denotes rejection of the null hypothesis. $^*$ Significant at 5% level.
Standard errors of the estimated parameters are in parentheses.
See the notes for Table 6.4.

**Table 6.9**
**Estimated TAR-GARCH Model For Log-price Changes of Crude Oil Futures And Test of Serial Dependence in Residuals**

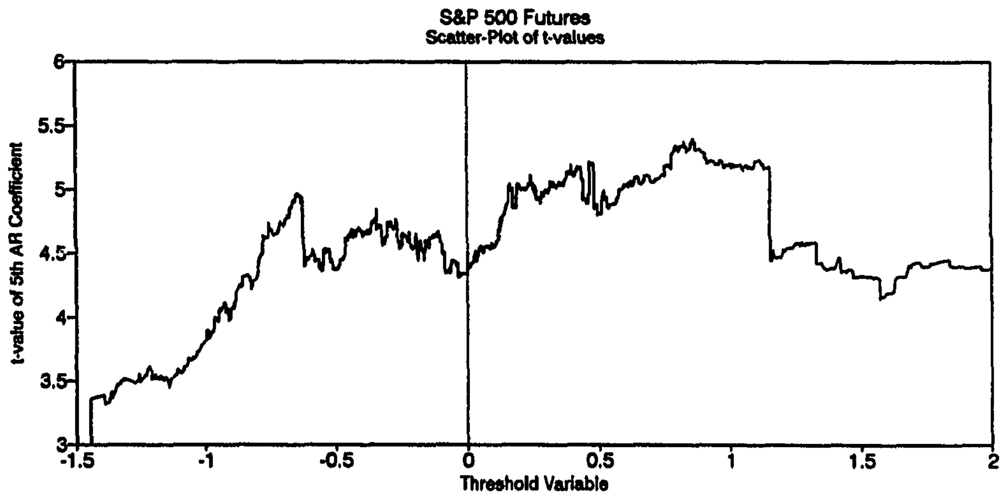
| | Region I ($x_{t-2}$<0.21) (Parameters with Dummy D) | Region II ($x_{t-2}$≥0.21) (Base Case) |
|---|---|---|
| c | | 0.0003 (0.062) |
| $\beta_1$ | | -0.006 (0.085) |
| $\beta_2$ | | -0.012 (0.075) |
| $\beta_3$ | | 0.006 (0.078) |
| $\beta_4$ | | 0.109 (0.076) |
| $\beta_H$ | | -0.011 (0.127) |
| $b^i_1$ | 0.044 (0.046) | -0.006 (0.034) |
| $b^i_2$ | -0.043 (0.055) | 0.006 (0.035) |
| $b^i_3$ | 0.044 (0.044) | -0.004 (0.033) |
| $b^i_4$ | 0.018 (0.043) | -0.014 (0.032) |
| $b^i_5$ | -0.068** (0.042) | 0.004 (0.027) |
| $b^i_6$ | -0.035 (0.044) | 0.020 (0.032) |
| $b^i_7$ | 0.026 (0.043) | -0.010 (0.031) |
| $b^i_8$ | -0.018 (0.036) | 0.021 (0.029) |
| $\alpha_0$ | | 0.025 (0.072) |
| $\alpha_1$ | | 0.142* (0.010) |
| $\alpha_2$ | | 0.858* (0.010) |
| $a_1$ | | 0.414* (0.128) |
| $a_2$ | | -0.523* (0.113) |
| $a_3$ | | 0.221* (0.087) |
| $a_4$ | | -0.095 (0.121) |
| $a_H$ | | 0.403* (0.120) |
| Test of Serial Dependence in Standardized Residuals | | |
| Q | 29.1 | |
| $Q^2$ | 25.2 | |
| BDS | -0.55 | |

* Significant at 5% level. ** Significant at 10% level.
Standard errors of the estimated parameters are in parentheses.
See the notes for Table 6.4 and Table 6.6.

Figure 6.1

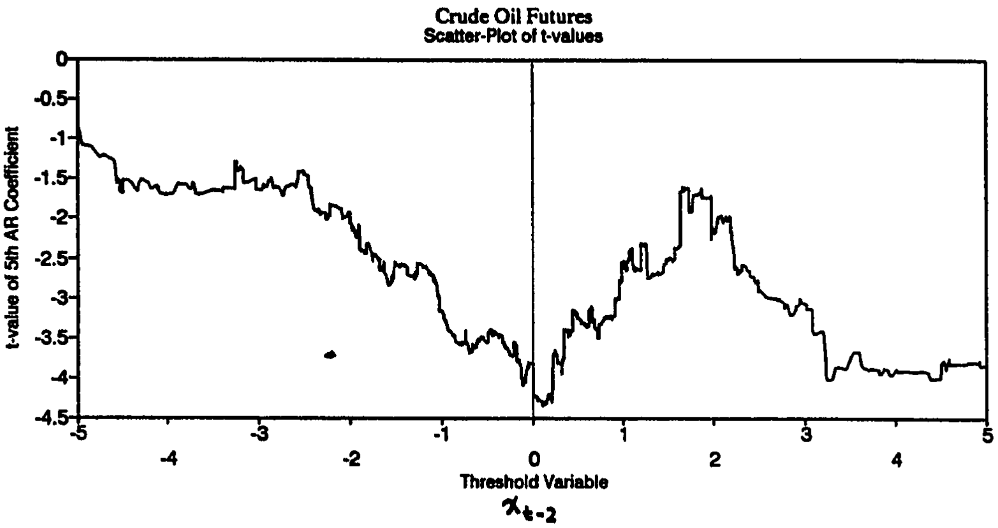


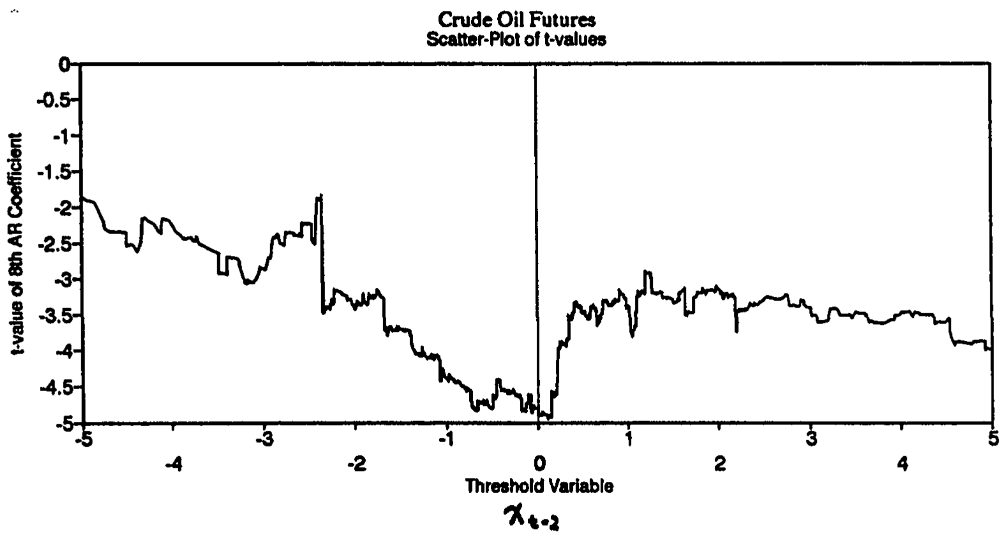

$x_{t-1}$




$x_{t-1}$

# Figure 6.2


Crude Oil Futures
Scatter-Plot of t-values
t-value of 5th AR Coefficient
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-4
-3
-2
-1
0
1
2
3
4
5
Threshold Variable
$x_{t-2}$


Crude Oil Futures
Scatter-Plot of t-values
t-value of 8th AR Coefficient
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
Threshold Variable
$x_{t-2}$

## 6.6 Summary

This chapter reports econometric findings based on price movements in S&P 500 futures, Crude Oil futures, Japanese Yen futures, Deutsche Mark futures, and Eurodollar futures over 1984-1993. Dickey-Fuller unit root tests indicate that the log-prices of all five futures are not stationary and have one unit root. Further, unit root tests show that the log-price changes of all five futures are stationary with no unit root. Therefore this suggests that we should use the log-price changes in the study of futures prices.

The five tests used in this chapter for detecting serial dependence in time series data are the Ljung-Box, $Q^2$, BDS , TAR-F, and Bispectral test. When studying the price movements of the five futures, the tests were applied to the log-price changes of the futures. The BDS test shows that log-price changes for all five futures are not IID. The Ljung-Box test indicates serial dependence in the log-price changes for the S&P 500, Crude Oil, and Eurodollar futures, but it can not detect serial dependence in log-price changes for the Japanese Yen and Deutsche Mark futures. The $Q^2$ test shows the serial dependence in squared log-price changes of all five futures. The TAR-F test detected the nonlinear serial dependence in the log-price changes of all five futures except the Japanese Yen futures. The Bispectral test found the nonlinear serial dependence in four of five futures with the exception being Crude Oil futures.

The first model of serial dependence in the log-price changes of these five futures was based on a linear AR model with the days-of-the-week effect and the holiday effect in conditional mean. The AR order for each futures is determined by Akaike information criterion. The residuals of the models can pass the Ljung-Box test, but they can not pass neither the $Q^2$ test nor the BDS test. They indicated that the serial dependence in log-price changes of the five futures in nonlinear. Because the value of the $Q^2$ statistic for the log-price changes is large, the serial dependence is probably the results of conditional heteroskedasticity. Therefor we applied the GARCH model to the log-price changes of futures.

The GARCH model specified has a linear AR process in conditional mean, where the conditional variance depends on the residual square and the variance of last period, as well as the days-of-the-week and holiday effect. The results show that the conditional variance has strong dependence on the sample variance and the conditional variance of last period. The results also suggest that the days-of-the-week and holiday effect in variance are significant. The standardized residuals of all five futures pass the Ljung-Box and $Q^2$ test. The standardized residuals of Japanese Yen futures, Deutsche Mark futures, and Eurodollar futures also pass the BDS test. We conclude that these three futures can be modeled by the GARCH process. But the standardized residuals of S&P 500 futures and Crude Oil futures failed to pass the BDS test. Thus the GARCH models are not suited for these two futures.

Because the large value of the TAR-F statistic is obtained for the log-price changes of S&P 500 futures and of Crude Oil futures, the TAR model appears to be a good alternative for modeling the log-price changes of S&P 500 futures and of Crude Oil futures. For the log-price changes of S&P 500 futures, the threshold 1 and three threshold regions are identified based on the arranged recursive regression,. The estimation of the TAR model also shows that the AR coefficients are different when the log-price change in last period falls into different threshold regions. The standardized residuals of the TAR model pass the Ljung-Box, $Q^2$, and BDS test. This shows the TAR model can adequately describe the log-price changes of S&P 500 futures.

For the Crude Oil log-price changes, the log-price change from 2 periods ago is selected as the threshold variable. Two threshold regions are identified. The estimation of the TAR model confirms that the AR coefficients are different when the log-price change from 2 periods ago falls into different threshold regions. The standardized residuals of the TAR model passed only the Ljung-Box test. They pass neither the $Q^2$ test nor the BDS test. Consequently the TAR model is not adequate for log-price change of Crude Oil futures. Because the value of the $Q^2$ statistic calculated for the log-price changes is very large, and the estimation of the GARCH model for the log-price changes of the Crude Oil shows significant serial dependence in the conditional variance, a combined TAR-GARCH model is fitted to the log-price changes of Crude Oil futures. The estimated results show that the AR coefficients in the conditional mean do have different values when the log-price change from 2 periods ago falls into different threshold regions. The results also show that the variance has strong dependence on the residual square and on the variance of last period, the days-of-the-week effect and the holiday effect in variance are significant. The standardized residuals of the combined TAR-GARCH model can pass the Ljung-Box test, the $Q^2$ test, and the BDS test. These test results indicated that the TAR-GARCH model can adequately describe the log-price changes of Crude Oil futures.

The results found in this chapter indicated serial dependence exists in the log-price changes of the five futures studied which reject the random walk model. The results also rejected the mean-reverting model because it requires the price changes over short horizon to follow a random walk. For price changes of Japanese Yen futures and of Deutsche Mark futures, the GARCH model do not show any significant serial dependence in the conditional mean. The GARCH model only has serial dependence in conditional variance. So the price changes of Japanese Yen futures and of Deutsche Mark futures are consistent with the martingale model. For the log-price changes of S&P 500 futures, of Crude Oil futures, and of Eurodollar futures, there is serial dependence in the conditional mean of the model describing them. Thus the price changes of these three futures do not follow the martingale model.

Our finding indicate that the log-price changes of three futures have conditional mean which depends on past log-price changes. This shows that current price can be predicated from past prices. But whether this is consistent with efficient market hypothesis remains to be investigated. Fama (1991), in a survey of recent research on the efficient market hypothesis, pointed out that the predictability of the price changes in financial markets does not necessarily reject the efficient market hypothesis. To test if the efficient market hypothesis is violated, we need to go further to see how much of the variation in price changes can be predicted by the model. If only a small part of price variation is predictable, then the prediction of prices probably can not be used for making a profit when there is a transaction cost of trading futures.

# CHAPTER 7

# CONCLUSIONS

Modeling price dynamics in financial markets has become an important research area in financial economics. In the past the empirical studies of financial price movements were based on models that were incapable of detecting or modeling nonlinearity and serial-dependence characterized financial market data. Because of the limitation of these models, methods used to study price dynamics in financial markets have gradually shifted from previous linear techniques to nonlinear techniques and models applicable to inherently nonlinear financial data. In this dissertation, three new statistical tests are studied and two econometric nonlinear time-series models are applied to futures prices. The tests and models studied are fully applicable to a nonlinear data generating process. The findings reported in this dissertation show that the tests and models are fundamentally useful for detecting and modeling nonlinear time-series process in financial prices.

This dissertation concerned and investigated the finite sample properties of the BDS test, the TAR-F test, and the $Q^2$ test. The investigation was based on Monte Carlo experiments. Monte Carlo experiments consisted of nine data generating processes (DGPs) which represent three IID time series, two linear processes, and four nonlinear time series processes. For each DGP, several parameter values and sample sizes were used to specify a Monte Carlo experiment. The results of Monte Carlo experiments have been summarized using response surfaces. The response surfaces are statistical summaries of the Monte Carlo findings. They show how the variation of the test statistics can be linked to experiment sample size and to variation in the parameters of the DGP. The main findings which emerge from the Monte Carlo investigation are as follows.

First, the finite sample rejection frequency under the null hypothesis for all three tests is quite close to the asymptotic probability. Second, under the alternative hypothesis, the power of the tests reaches unity at sample size of 1000 when the DGPs of alternative hypothesis are not too close to the DGP of null hypothesis. A DGP's departure from DGP of null hypothesis is reflected by the parameter value set in the Monte Carlo experiment. For example, the power of the tests is nearly 100% at sample size of 1000 when the parameter of DGP of nonlinear MA is 0.3. If the departure of DGPs from the DGP assumed under the null hypothesis is larger, the power of the tests can reach unity at smaller sample size. If the sample size is larger, the power of the tests can reach unity when the departure of DGP from the DGP assumed under the null hypothesis is smaller. Finally, if either one or both the departure of DGP from the DGP assumed under the null hypothesis and the sample size are smaller, the power of these tests will be much less than unity.

Overall, the finite sample performance of the three tests can be summarized as follows. The BDS statistic has higher power than the $Q^2$ statistic does against the DGPs of linear time series processes. The TAR-F statistic has highest power among the three against DGPs of nonlinear time series processes except when DGP is GARCH. In this case, it appears that the BDS statistic has the highest power. The Monte Carlo experiments indicated that the BDS test yields the most reliable results with L=1.5 (correlation length) and M=3 (embedding dimension). The results here are important to researchers because the number of the BDS statistics to be calculated can be cut down and we do not have to look at many BDS statistics calculated and decide which one to use for hypothesis test. The Monte Carlo experiments also showed that the TAR-F statistic can be used to identify the threshold lag of a TAR process. This result is also very useful for TAR model building. We used this finding in modeling futures prices in Chapter 6 of this dissertation.

The Monte Carlo findings of three statistical tests in this dissertation may provide information useful for researchers building econometric model of financial markets. First, the three tests appear to be quite reliable for detecting varieties of serial dependence time series data. The tests also appear to be able to detect nonlinear serial dependence. Linear testing methods sometime fail in this case. Second, the ideal working range of these tests is for sample size of 1000 or larger and the time series being tested not too close to the null hypotheses of the tests. If the sample size is small, or the time series is too close to the null hypothesis, the tests will have low power to reject the false null hypothesis. We have used our findings in an empirical application.

Specifically in Chapter 6, the finite sample properties of these three new statistical tests were applied to analyze futures prices. In this dissertation, I also applied the linear technique of Ljung-Box autocorrelation method and the nonlinear technique of the Bispectral test to futures prices. Empirical results were based on a study of five futures prices. Our findings show that futures prices have nonlinear serial dependence. The linear technique of Ljung-Box autocorrelation test can fail to detect nonlinear serial dependence in futures prices. The Bispectral test gave similar test results as the three tests did. The econometric linear time series models are inadequate for modeling futures prices. The futures prices of Japanese Yen, Deutsche Mark, and Eurodollar have autoregressive conditional heteroskedasticity (ARCH) type nonlinear serial dependence and they can be modeled by nonlinear model of GARCH process. The S&P 500 futures prices appear nonlinear in the conditional mean. Accordingly the series is modeled using a TAR model. In contrast, the futures prices of Crude Oil have ARCH nonlinearity and nonlinearity in conditional mean, so they have to be modeled by a combined TAR-GARCH model.

The study of price movements of five futures in this dissertation shows that price movements do not conform to a random walk, rather they appear have some serial dependence. The results here also rejected the mean-reverting model because this model requires price changes to follow a random walk in the short-run. Japanese Yen futures and Deutsche Mark futures conform to a martingale model because there is no significant serial dependence in the conditional mean of their log price changes. For S&P 500 futures, Crude Oil futures, and Eurodollar futures, the conditional mean of log price changes has serial dependence. Thus they do not conform with the martingale model. But whether the serial dependence found in the conditional mean of these futures prices can be used for formulating profitable trading strategy in violation of the efficient market hypothesis remains to be investigated.

The following conclusion emerged from the study of futures prices in this dissertation. When analyzing price changes in futures markets, we must account for the nonlinear serial dependence. To model futures prices, we need to use nonlinear models rather than linear models because the underlying data are inherently nonlinear. Furthermore, we should not restrict the use of nonlinear models to those with conditional heteroskedasticity (eg., GARCH model), because some futures appear to have nonlinear conditional mean. This recommends the use of TAR and TAR-GARCH model. The findings also pointed out the need for more research on forecasting conditional volatility based on models with conditional heteroskedasticity and testing of the efficient market hypothesis with nonlinear models.

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